An Algebraic Approach to the Dirac Equation

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We begin with a traditional derivation of the Dirac Equation, and proceed to study and motivate it from an algebraic perspective using the representations of the Lorentz Algebra. We close by describing possible ways to extend the Dirac Equation to higher spins.

I. A TRADITIONAL DERIVATION OF THE DIRAC EQUATION

A. Factoring the Relativistic Hamiltonian [1]

In the non-relativistic case, we have the Schrödinger Equation for a free particle [1]

Theorem I.1 (**Nonrelativistic Schrödinger Equation**).

$$i\hbar\frac{\partial\left|\psi\right\rangle}{\partial t} = \frac{P^2}{2m}\left|\psi\right\rangle \tag{1}$$

In the relativistic case, we can use the relativistic free particle Hamiltonian [11].

$$H = (c^2P^2 + m^2c^4)^{\frac{1}{2}} \tag{2}$$

and thus, try to solve the relativistic free-particle Schrödinger Equation

$$i\hbar\frac{\partial\left|\psi\right\rangle}{\partial t} = (c^2P^2 + m^2c^4)^{\frac{1}{2}}\left|\psi\right\rangle \tag{3}$$

It isn't obvious how to solve this, but as is often the case, we begin by making a leap of faith. Suppose that the expression inside the square root can be written as a perfect square [12]. Then, we assume that this square root is of the form

$$c^2P^2 + m^2c^4 = (c\alpha_xP_x + c\alpha_yP_y + c\alpha_zP_z + \beta mc^2)^2 \tag{4}$$
$$= (c\alpha\cdot\mathbf{P} + \beta mc^2)^2 \tag{5}$$

We can match both sides of the equation (omitting computation) to yield the following relations on α and β

$$\alpha_i^2 = \beta^2 = 1 \tag{6}$$
$$\{\alpha_i, \alpha_j\} = 0 \tag{7}$$
$$\{\alpha_i, \beta\} = 0 \tag{8}$$

where $\{\cdot,\cdot\}$ represents the anticommutator. Clearly α and β aren't numbers. They must be Hermitian, traceless, and have eigenvalues ± 1. We require that none of the matrices are equal because the anticommutator of a nonzero matrix with itself cannot be 0. The smallest matrices that work are 4x4 matrices. One such set is

$$\alpha = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{9}$$

which are referred to as the Gamma matrices. This gives us the Dirac Equation.

Theorem I.2 (**Dirac Equation**).

$$i\hbar\frac{\partial\left|\psi\right\rangle}{\partial t} = (c\alpha\cdot\mathbf{P} + \beta mc^2)\left|\psi\right\rangle \tag{10}$$

where α and β are as above.

B. Time-Space Symmetry [1]

We can rewrite the Dirac Equation by using the four-gradient.

$$i\hbar\frac{\partial\left|\psi\right\rangle}{\partial t} = (c\alpha\cdot\mathbf{P} + \beta mc^2)\left|\psi\right\rangle \tag{11}$$
$$\implies ic\hbar\partial_0\left|\psi\right\rangle = \sum_{j=1,2,3} -ic\hbar\alpha_j\cdot\partial_i + \beta mc^2\left|\psi\right\rangle \tag{12}$$
$$\implies i\hbar\alpha_\mu\partial_\mu\left|\psi\right\rangle - mc\left|\psi\right\rangle = 0 \tag{13}$$

where in the second equation we used $\partial_0 = \frac{1}{c}\partial_t$

We can now see that the Dirac Equation is symmetric in time and space, first order in both, a desirable property for a relativistic quantum equation.

C. The Dirac Equation Describes Spin-$\frac{1}{2}$ Particles [2]

The next step is figuring out what sort of states this equation describes. Define the Dirac Hamiltonian as

$$H_D = c\alpha\cdot\mathbf{P} + \beta mc^2 \tag{14}$$

In order to better understand H_D, we search for operators that commute with H_D, as these correspond to time-conserved properties of the state.

Proposition I.1. The total angular momentum $\mathbb{L} + \mathbb{S}$ commutes with the Dirac Hamiltonian H_D if we use spin-$\frac{1}{2}$ operators.

Proof. For starters, let's look at the orbital angular momentum $\mathbb{L}$. For the sake of making computation easier, we choose units such that $c = \hbar = 1$ and use the Einstein convention. Noting that $\beta = I$,

$$\begin{aligned}
[L_j, H_D] &= [L_j, \alpha \cdot \mathbf{P}] && (15)\\
&= [\epsilon_{jkl} r_k p_l, a_i p_i] && (16)\\
&= a_i [\epsilon_{jkl} r_k p_l, p_i] && (17)\\
& && (18)
\end{aligned}$$

Now, because $[r_i, p_j] = i\delta_{ij}$, we have

$$\begin{aligned}
\alpha_i [\epsilon_{jkl} r_k p_l, p_i] &= \alpha_i \epsilon_{jkl} (r_k p_l p_i - p_i r_k p_l) && (19)\\
&= \alpha_i \epsilon_{jkl} (i\delta_{ik} p_l + p_i r_k p_l - p_i r_k p_l) && (20)\\
&= \alpha_i \epsilon_{jil} p_l && (21)\\
\implies [L_j, H_D] &= \epsilon_{jil} \alpha_i p_l && (22)
\end{aligned}$$

We're a little disheartened, because this means that orbital angular momentum is not conserved here, but we proceed onwards, and check commutatitivity with the spin operators S. Using the marvelous power of hindsight, we elect to try spin-$\frac{1}{2}$ operators first.

However, it isn't clear how to construct spin-$\frac{1}{2}$ operators that operate on four-dimensional space.

Lemma I.1. The operators

$$S_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \tag{23}$$

are spin-$\frac{1}{2}$ operatorso that operate on four-dimensional space, where each entry of the above matrix is $2x2$ and σ_i are Paulli matrices.

Proof. Any such triple of operators $\{S_i\}$ must satisfy

$$[S_i, S_j] = iS_k \tag{24}$$

and

$$\mathbb{S}^2 = S_x^2 + S_x^2 + S_z^2 = l(l+1) = \frac{3}{4} \tag{25}$$

We know that the operators $\frac{1}{2}\sigma_i$ satisfy these conditions over a 2-dimensional space. Thus, by matrix block multiplications, the matrices

$$S_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \tag{26}$$

satisfy the above relations as well, but operate over four-dimensional space. □

Now, we can check the commutator with the Hamiltonian. Noticing that we can pull the p_i out

$$\begin{aligned}
[S_j, H_D] &= [S_j, \alpha_i p_i] + m[S_j, I] && (27)\\
&= p_i [S_j, \alpha_i] && (28)\\
&= p_i \epsilon_{jik} \alpha_k && (29)\\
&= \epsilon_{jil} \alpha_i p_l && (30)\\
\implies [S_j, H_D] + [L_j, H_D] &= [L_j + S_j, H_D] = 0 && (31)
\end{aligned}$$

And thus, the total angular momentum $\mathbb{L} + \mathbb{S}$ is conserved, so the Dirac Equation does indeed describe spin-$\frac{1}{2}$ particles, as desired. □

II. PRELIMINARIES

The above derivation was short, but it was not particularly motivated. In particular, the system of constraints (equations 6-8) that we solved do not relate to relativity in any obvious way. In order to provide some more physical intuition for what's going on, we'll show that the Dirac Equation and the four-component representation for an electron is a result of Lorentz Symmetry. We'll develop some mathematical machinery along the way. For readers who've seen some algebra before, please excuse the lack of rigor in some of the definitions and derivations presented.

A. Lorentz Transformations

In nonrelativistic mechanics, we require that the laws of physics be invariant under Galilean Transformations, which preserve the Euclidean norm

$$x^2 + y^2 + z^2 \tag{32}$$

Analogously, in relativistic mechanics, we require that the laws of physics be invariant under **Poincaré transformations**, which are continuous transformations characterized by their preservation of the Minkowski norm

$$(ict)^2 + x^2 + y^2 + z^2 = -c^2t^2 + x^2 + y^2 + z^2 \tag{33}$$

For the purposes of this paper, we'll restrict our attention to **Lorentz Transformations**, which are are Poincaré transformations that are not translations. [13]

Lorentz transformations come in two types - rotations and **boosts**, which are (rather unhelpfully) defined as Lorentz transformations that are not rotations. Our task is now to understand Lorentz Transformations more formally.

B. A Crash Course in Representation Theory [3]

Definition II.1. A **group** G is a (possibly infinite) set of elements and a rule for multiplying elements that takes two elements $g_1, g_2 \in G$ and outputs $g_3 \in G$. In addition, there's an identity element e, and every element has an inverse such that $gg^{-1} = e$.

The set of rotations in three-dimensions is a group, because we can compose any two rotations to create a third (this isn't obvious but it's true). Lorentz transformations also form a group, as we can compose any two norm-preserving transformations to create a third. In physics, we often care about **matrix groups**, where group elements are invertible matrices and multiplication is the usual matrix multiplication.

Definition II.2. A **group representation** is a map R from group elements to matrices such that if $g_1 g_2 = g_3$ then $R(g_1)R(g_2) = R(g)$.

Intuitively, we assign a matrix to every group element such that the matrices satisfy the same multiplication table as the group elements themselves. Of course, all the matrices are square (invertible) and act on the same vector space (otherwise they couldn't be multiplied). The dimension of a representation is the dimension of the vector space on which it acts. Another way to think about this is that we map a group to a matrix group.

As an example, consider the group of rotations. We always have the **trivial representation** which takes every element in the group to 1. In addition, we have the group of three-dimensional rotation matrices, which by definition satisfy the same multiplication table as the group of rotations and thus form a three-dimensional representation of the rotation group. The rotation matrices are the orthogonal matrices with determinant 1, and this matrix group is called $SO(3)$, the special orthogonal group [14]

Similarly, the Lorentz group, which is the same group extended by one more dimension, can be represented by the matrix group $SO(3,1)$, where $(3,1)$ comes from the sign change in the Minkowski norm. To be precise, these are matrices Λ such that

$$\Lambda \eta \Lambda^\dagger = I \tag{34}$$

where

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

Given two representations $R_1 : G \to V$ and $R_2 : G \to W$, we can write a third representation R_3 where

$$R_3(g) = \begin{bmatrix} R_1(g) & \mathbf{0} \\ \mathbf{0} & R_2(g) \end{bmatrix} \tag{36}$$

Thus, the matrices in R_3 are block diagonal, and thus satisfy the requisite multiplication laws because R_1 and R_2 do.

Definition II.3. A representation is **reducible** if there exists a basis in which every matrix of the representation is block diagonal with the same block sizes, and **irreducible** if no such basis exists. [15]

If the set of irreducible representations of a group are known, then all possible representations can be constructed by the process we described above [16].

C. Lies, Damn Lies [4]

Definition II.4. A **Lie Group** is a group that is also a smooth manifold. For our purposes, it's sufficient to say that most matrix groups are Lie Groups [17], and in particular, both $SO(3)$ and $SO(3,1)$ are Lie Groups.

Frequently, we can study Lie Groups by studying the corresponding **Lie Algebra**. We'll restrict ourselves to the study of matrix lie algebras.

Definition II.5. The **Lie Algebra** $\mathfrak{g}$ of a matrix group G is the set of all matrices X such that $e^{itX} \in G \forall t \in \mathbb{R}$, together with a **Lie Bracket**, a bracket operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties

- $[\cdot,\cdot]$ is **anti-symmetric**, so

$$[X, Y] = -[Y, X] \forall X, Y \in \mathfrak{g} \tag{37}$$

- $[\cdot,\cdot]$ is **bilinear**, so for all $a, b \in \mathbb{C}$ and $X, Y, Z \in \mathfrak{g}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z] \tag{38}$$
$$[X, aY + bZ] = a[X, Y] + b[X, Z] \tag{39}$$

- $[\cdot,\cdot]$ satisfies the **Jacobi Identity**

$$[X, [Y, Z] + [Y, [Z, X] + [Z, [X, Y]] = 0 \forall X, Y, Z \in \mathfrak{g} \tag{40}$$

The elements of the Lie algebra are said to be **generators** for the corresponding Lie group. For a matrix lie algebra, the bracket is the **commutator**, which we can check has all the properties of the Lie Bracket.

For the Lie Group of rotations SO_3, we write the Lie algebra $\mathfrak{so}(3)$. As we've seen, the Pauli matrices form a basis for this Lie Algebra, with the correspondence

$$R(\hat{n}, \alpha) = e^{-\frac{i\alpha(\sigma\cdot\hat{n})}{2}} \tag{41}$$

The Pauli Matrices also satisfy the appropriate bracket relations using the commutator.

We will also mention briefly a useful result from linear algebra.

Proposition II.1. For any complex matrix X,

$$e^{\text{trace}(X)} = \det e^X \tag{42}$$

Proof. Consider the eigenvalues and eigenvectors of X. Because the matrix exponenetial is just a power series, any eigenvector of X is also an eigenvector of e^X. Furthermore, if $Xv = \lambda v$, then,

$$e^X v = (\sum \frac{X^n}{n!})v = \sum \frac{\lambda^n}{n!} v = (\sum \frac{\lambda^n}{n!})v = e^\lambda v \tag{43}$$

Then, remembering that the trace is the sum of the eigenvalues and that the determinant is the product of the eigenvalues,

$$e^{\text{trace}(X)} = e^{\sum_i \lambda_i} = \prod_i e^{\lambda_i} \tag{44}$$

and we're done. □

D. Clifford Algebras

For our purposes, a Clifford Algebra is a set of four matrices γ satisfying the constraints in Equations 6-8. We can write these constraints more concisely.

Definition II.6. A Clifford Algebra is a set of four matrices γ satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu,\nu} \tag{45}$$

where $\mu, \nu \in 0, 1, 2, 3$ and $\eta_{\mu,\nu}$ refers to the μ, ν entry.

Our ultimate goal is to understand the connection between Clifford Algebras, which arise naturally by taking the square root of the relativistic Hamiltonian (Equation 2) and Lorentz symmetries.

III. PUTTING IT ALL TOGETHER [5] [6]

A. Generators of the Lorentz Algebra

We are interested in studying $SO(3,1)$. In particular, we'd like to study its matrix representations. As is often the case when studying representations of Lie Groups, it is frequently easier to study the representations of the underlying Lie Algebra, because we can then obtain a representation of the Lie Group simply by exponentiating. This is to say, we'd like to find matrices that when exponentiated, can generate arbitrary Lorentz transformations.

One such representation, and perhaps the simplest, is the **vector representation**. The proof of how this representation is derived is left as a footnote [18]

We index the six generators as $V_{\mu\nu}$ where $\mu, \nu \in \{0, 1, 2\}$. The utility of this will become clear in a moment.

$$V_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} V_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$V_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} V_{01} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{46}$$

$$V_{02} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} V_{03} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Now, let the three antisymmetric matrices be J_i, and the three symmetric matrices be K_i. Then, we can check that

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{47}$$

$$[K_i, K_j] = i\epsilon_{ijk} K_k \tag{48}$$

$$\tag{49}$$

Exponentiating the J_i shows that they correspond to 3D rotations. For example,

$$e^{J_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(1) & -\sin(1) & 0 \\ 0 & \sin(1) & \cos(1) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

The other three generators correspond to boosts, and also satisfy a nice commutation relation. It turns out that these commutation relations must hold for any representation of the Lie Algebra [19] .

We can condense the commutation relations into a single expression below, where each $J_{..}$ is a matrix and $\eta_{..}$ is an entry of η.

Proposition III.1 (Lie Bracket of Lorentz Algebra).

$$[V_{\mu\nu}, V_{\rho\sigma}] = i(\eta_{\nu\rho} V_{\mu\rho} - \eta_{\mu\rho} V_{\nu\sigma} - \eta_{\nu\sigma} V_{\mu\rho} + \eta_{\mu\sigma} V_{\nu\rho} \tag{51}$$

In conclusion, we've shown that for any matrix representation of the Lie Algebra, there must be a way to assign indices to the matrices such that the above commutation relation holds.

B. Clifford Algebras and Lorentz Algebras

We will now show a relationship between Clifford Algebras and representations of the Lorentz Group.

Consider the matrices

$$S_{\mu,\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{52}$$

We claim that the matrices $S_{\mu\nu}$ form a representation of the Lorentz Lie Algebra, which is to say that they satisfy the commutation relations in Equation 44.

Lemma III.1.

$$[S_{\mu\nu}, \gamma_\rho] = \gamma_\mu \eta_{\nu\rho} - \gamma_\nu \eta_{\rho\mu} \tag{53}$$

Proof.

$$[S_{\mu\nu}, \gamma_\rho] = \frac{1}{2}[\gamma_\mu\gamma_\nu, \gamma_\rho] \tag{54}$$
$$= \frac{1}{2}\gamma_\mu\gamma_\nu\gamma_\rho - \frac{1}{2}\gamma_\rho\gamma_\mu\gamma_\nu \tag{55}$$
$$= \frac{1}{2}\gamma_\mu\{\gamma_\nu, \gamma_\rho\} - \frac{1}{2}\gamma_\mu\gamma_\rho\gamma_\nu - \frac{1}{2}\{\gamma_\rho, \gamma_\mu\}\gamma_\nu + \frac{1}{2}\gamma_\mu\gamma_\rho\gamma_\nu \tag{56}$$
$$= \gamma_\mu\eta_{\nu\rho} - \gamma_\nu\eta_{\rho\mu} \tag{57}$$

□

Proposition III.2. The matrices $S_{\mu\nu}$ form a representation for the Lorentz Lie Algebra, so

$$[S_{\mu\nu}, S_{\rho\sigma}] = \eta_{\nu\rho}S_{\mu\sigma} - \eta_{\mu\rho}S_{\nu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\nu\sigma}S_{\mu\rho} \tag{58}$$

Proof. Let's take $\rho \neq \sigma$. Then, using our earlier Lemma,

$$[S_{\mu\nu}, S_{\rho\sigma}] = \frac{1}{2}[S_{\mu\nu}, \gamma_\rho\gamma_{sigma}] \tag{59}$$
$$= \frac{1}{2}[S_{\mu\nu}, \gamma_\rho]\gamma_\sigma + \frac{1}{2}\gamma_\rho[S_{\mu\nu}, \gamma_\sigma \tag{60}$$
$$= \frac{1}{2}\gamma_\mu\gamma_\sigma\eta_{\nu\rho} - \frac{1}{2}\gamma_\nu\gamma_\sigma\eta_{\rho\mu} + \frac{1}{2}\gamma_\rho\gamma_\mu\eta_{\nu\sigma} \tag{61}$$

We have by definition that

$$\gamma_\mu\gamma_\sigma = 2S_{\mu\sigma} + \eta_{\mu\sigma} \tag{62}$$

and plugging this in yields the desired expression.

The matrices in Equation 9 yield the following matrices. We can check that they satisfy the commutation relation from Equation 48.

$$S_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} S_{13} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$S_{23} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} S_{01} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{63}$$

$$S_{02} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} S_{03} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We've found a representation of the Lorentz Lie Algebra that we know corresponds in some way to spin-$\frac{1}{2}$, but we'd like to understand this more formally. □

C. Representations of the Lorentz Group

Let's study the representations of the Lorentz Algebra. Remember that the J_i generate rotations and the K_i generate boosts. Now, take the linear combinations

$$J_i^+ = \frac{1}{2}(J_i + iK_i) \tag{64}$$
$$J_i^- = \frac{1}{2}(J_i - iK_i) \tag{65}$$
$$\tag{66}$$

These satisfy the commutation relations

$$[J_i^+, J_j^+] = i\epsilon_{ijk}J_k^+ \tag{67}$$
$$[J_i^-, J_j^-] = i\epsilon_{ijk}J_k^- \tag{68}$$
$$[J_i^+, J_j^-] = 0 \tag{69}$$

Thus, choosing these combinations gives us two commuting subgroups of the Lorentz algebra, each of which is isomorphic to $\mathfrak{so}(3)$. Commuting subgroups are intuitively nice because they behave "independently". More formally, if $X, Y \in \mathfrak{so}(3)$ commute,a

Lemma III.2.

$$[X, Y] = 0, X, Y \in \mathfrak{so}(3) \implies e^X e^Y \tag{70}$$

Proof. Write out the power series and watch things cancel. □

Proposition III.3.

$$\mathfrak{so}(3,1) = \mathfrak{so}(3) \times \mathfrak{so}3 \tag{71}$$

Proof. The J_i^+ and J_i^- each form subgroups of the Lie Algebra that commute with each other (per Equation 66). The lemma thus tells us transforming first by a a transformation from J_i^+ and then by a transformation from J_i^- (or vice-versa - the order is irrelevant) is the same as doing both at once. We know that any transformation is a combination of a boost and a rotation, and we now know that we can specify the two independently (this is not obvious, and it isn't generally true). □

Thus, we can describe representations of $\mathfrak{so}(3,1)$ by specifying two representations of $\mathfrak{so}(3)$ and "gluing" the corresponding vector spaces together[20].

As we know, the representations of $\mathfrak{so}(3)$ are Pauli matrices, and in particular we can specify any representation by an integer j. Thus, by specifying two numbers A and B, one for each of the subgroups, we get a representation.

D. The Weyl Representation

Let's look at irreducible representations of the Lorentz algebra containing spin-$\frac{1}{2}$. As it turns out, there are two: $\frac{1}{2}$: $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. Their structure is in fact quite simple. In $(\frac{1}{2},0)$,

$$J_i^+ = \frac{\sigma_i}{2} \tag{72}$$

$$J_i^- = 0 \tag{73}$$

wich works because we know the Paulli matrices satisfy the requisite commutator relations. Similarly, $(0,\frac{1}{2})$ is given by

$$J_i^+ = 0 \tag{74}$$

$$J_i^- = \frac{\sigma_i}{2} \tag{75}$$

We know that the J_i is a spin-$\frac{1}{2}$ representation by Equation 24.

Thus, for the actual rotation and boost generators, we can say

$$(\frac{1}{2},0)|J_i = \frac{\sigma_i}{2}, K_i = -i\frac{\sigma_i}{2} \tag{76}$$

$$(0,\frac{1}{2})|J_i = \frac{\sigma_i}{2}, K_i = i\frac{\sigma_i}{2} \tag{77}$$

$$\tag{78}$$

It's clear that these are both representations of the Lorentz Group. Now, let's take their direct sum (see Equation 35), which we write as $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. This gives us the matrix representation

$$S_{12} = \begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix} S_{13} = \begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$$

$$S_{23} = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} S_{01} = \begin{bmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \tag{79}$$

$$S_{02} = \begin{bmatrix}0&-1&1&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} S_{03} = \begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$$

and these are the same matrices as we found in Equation 60!

E. What this all means

The Dirac Equation, which is traditionally derived by starting with the square root of the relativisitc Hamiltonian, requires finding a Clifford Algebra representation. We've shown that every Clifford Algebra representation can be used to construct a representation of the Lorentz Algebra. In particular, the canonical 4x4 representation corresponds to the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz Group.

IV. EXTENSIONS

At this point, it seems as if we are done. We've found an extremely powerful equation relating special relativity and quantum mechanics for a spin-$\frac{1}{2}$ particle, and seem to understand it fairly well. By approaching its derivation and structure from an algebraic perspective, we understand the nature of the solutions and the particular representation of the Lorentz group that yield the Dirac Equation.

Now, we'll briefly look at an extensions that we can make, based on the groundwork that we've laid. As a caveat, some of the work below is my own, and is likely to have errors. Any comments on them would be appreciated.

A. Higher-Order Clifford Representations [10]

We can immediately take the direct sum of the gamma matrices of equation 9 to get a set of 8x8 matrices that also satisfy the Clifford Algebra relations [21].

$$\alpha_i = \begin{bmatrix}0&\sigma_i&0&0\\\sigma_i&0&0&0\\0&0&0&\sigma_i\\0&0&\sigma_i&0\end{bmatrix} \beta = \begin{bmatrix}I&0&0&0\\0&-I&0&0\\0&0&I&0\\0&0&0&-I\end{bmatrix} \tag{80}$$

In fact we can do this to get representation of dimension $4k$ for any k [22]. It isn't clear, however, that these representations correspond to Lorentz representations. For example, we might think that the 8x8 gamma matrices apply to spin-$\frac{3}{2}$ particles, using the spin operators

$$S_i = \begin{bmatrix}\sigma_i&0\\0&\sigma_i\end{bmatrix} \tag{81}$$

where the spin operators are

$$\sigma_1 = \begin{bmatrix}0&\sqrt{3}&0&0\\\sqrt{3}&0&2&0\\0&2&0&\sqrt{3}\\0&0&\sqrt{3}&0\end{bmatrix} \tag{82}$$

$$\sigma_2 = \begin{bmatrix}0&-i\sqrt{3}&0&0\\i\sqrt{3}&0&-2i&0\\0&2i&0&-i\sqrt{3}\\0&0&i\sqrt{3}&0\end{bmatrix} \tag{83}$$

$$\sigma_3 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \tag{84}$$

If this worked, we could get a Dirac-like equation for spin-$\frac{3}{2}$ particles by following an approach as in section 1. However, this fails when we try to show that the Hamiltonian based on the 8x8 gamma matrices commutes with spin-$\frac{3}{2}$ angular momentum, as the commutator with $\mathbb{L}$ is the same as in Equation 22 but the commutator with $\mathbb{S}$ doesn't have a nice form.

However, now that we're armed with algebraic tools, we can try a more sophisticated approach. We can find the representation of the Lorentz algebra that corresponds to these gamma matrices, as per Equation 49. We know that one must exist per Lemma III.2. However, as I'm both out of space and out of time, I'll leave this avenue unexplored [23]

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[1] R. Shankar *Principles of Quantum Mechanics, 2nd Ed.* (Plenum Press, 1994).
[2] J. Franklin *Quantum Mechanics I - Lecture 35: Relativistic Quantum Mechanics II* (Reed College, 2008)
[3] M. Artin *Algebra, 2nd Ed.* (Prentice Hall, 2011)
[4] J. H. Yoo *Lie Groups, Lie Algebras, and Applications in Physics* (University of Chicago, 2015)
[5] M. Schwartz *QFTI - Lecture 10: Spinors and the Dirac Equation* (Harvard College, 2010)
[6] D. Tong *The Dirac Equation* (University of Cambridge, 2012)
[7] L. Lerner *Derivation of the Dirac Equation from a relativistic representation of spin* (European Journal of Physics Vol 17 No. 4, 1995)
[8] J. Maciejko, *Representations of Lorentz andd Poincaré Groups* (CMITP and Department of Physics, Stanford University)
[9] E. Cartan *The Theory of Spinors* (Dover Books on Mathematics, 2012)
[10] V. S. Varadarajan *Spinors* (UCLA)
[11] We won't derive this because it isn't particularly important for the rest of the paper, and because a derivation would probably take me over the page limit. In brief though, we construct a Lorentz-invariant action, find the Lagrangian based on the action, and get the Hamiltonian from the Lagrangian
[12] This is generally true. Any positive semidefinite operator has a square root - it is the operator with the same set of eigenvectors where eigenvalue λ is replaced with eigenvalue $\sqrt{\lambda}$. Thus, any bounded Hamiltonian can have a square root (if we choose our zero energy such that the operator is positive semidefinite)
[13] I don't know that much about how to deal with translations, but the general idea is that once we fix a momentum (infinitesimal translations are generated by momentum), we can look just at the group that leaves momentum fixed. I think this is called Wigner's Little Group method, and this allows us to classify massive particles in some nice way
[14] The word "special" corresponds to the determinant 1 condition. We know that the determinant must be ± 1 because it's orthogonal. The 3D orthogonal matrices with determinant -1 correspond to reflections, which aren't relevant here
[15] There's complications in the theory of infinite-dimensional representations, but we'll restrict ourself to the finite-dimensional case
[16] To formalize this notion, at least for finite groups and finite-dimensional representations, we use Maschke's Theorem. I'm not sure what the formalism is for infinite groups
[17] Any closed subgroup of $GL_n(\mathbb{C})$ is a Lie Group
[18] First, we show that the dimension of the Lie Algebra (the number of generators) is 6. This proof will assume general familiarity with algebra. We'll use the following lemma. It isn't hard to prove - for a proof, see an introductory text in algebra, like Artin.

$$X* = -X \iff (e^X)^\dagger e^X = I$$

Now for proving the actual result. First, note that matrices in $SO(3,1)$ are unitary with respect to some inner product (the one where the matrix of the form is η) and thus any exponential generators must be skew-Hermitian with respect to this form per our Lemma. Then, in any four-dimensional matrix representation of the generators, because we are looking for trace 0 skew-Hermitian matrix, we can count the number of degrees of freedom. Remember that all the entries in our generator are real. Then, the skew-Hermitian condition becomes an anti-symmetry condition, so $X_{ij} = -X_{ji}$ and therefore elements along the diagonal are 0. Then, the remaining degrees of freedom come from half of the remaining elements, of which there are 6. The explicit construction of the matrices is then basically just assigning one generator per entry, but signs get switched around because of the -1 in the Minkowski metric.
[19] It isn't hard to see that at least these commutation relations must hold in any other matrix representation. I think it's harder to show that there aren't other relations that must also hold. I think this corresponds to the faithfulness of this representation, which probably isn't too hard to work out, but I won't show it here.
[20] Formally, we can take the direct sum of two representations of $\mathfrak{so}(3)$ to get a representation of $\mathfrak{so}(3,1)$, and in finite dimensions a direct sum is the same as a product
[21] because we can operate on block diagonal matrices by operating on each block individually, and each block satisfies the relations
[22] We won't prove this, but I think it's actually the case that the **only** satisfying representations are of dimension $4k$. I don't understand the subtleties and caveats of this well enough to explain this, but I'm pretty sure it's true
[23] I have discovered a truly marvelous proof of this, which this paper is too short to contain :)