COMPUTING HILBERT MODULAR FORMS OF NONPARITIOUS WEIGHT

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Abstract. We design and implement an algorithm for computing q-expansion bases of spaces of Hilbert modular forms of nonparitious weight over fields of narrow class number 1. We use this algorithm to compute spaces of Hilbert modular forms over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ of weight $(1,2)$ with Galois stable levels of norm up to 1500 and quadratic nebentypus. To study this algorithm, we introduce the "elemental" Hecke algebra, a finite algebra generated by rescalings of the usual Hecke operators acting on a space of Hilbert modular forms. The elemental Hecke algebra is equivalent to the usual Hecke algebra in paritious weight, but retains certain rationality properties even in the nonparitious setting even when the usual Hecke algebra is poorly behaved. Using the elemental Hecke algebra, we are also able to present self-contained proofs of some standard facts about Hilbert modular forms that we use in the algorithm.

1. Introduction

The Langlands programme is a sweeping web of conjectures relating automorphic forms, Galois representations, motives, and L-functions. Typically, given an object in one of these four worlds, one hopes to construct corresponding objects in the others. For example, the H^1 of an elliptic curve $E/\mathbb{Q}$ is a motive to which one can associate a compatible family of ℓ-adic Galois representations (the first étale cohomology of E), an L-function, and most nontrivially, an automorphic form. However, not every automorphic form is expected to contribute directly to this story. An automorphic representation over a field K is said to be L-algebraic if its archimedean components satisfy a certain integrality condition. Only L-algebraic automorphic representations are expected ([BG14, Conjecture 3.21]) to have associated compatible systems of ℓ-adic Galois representations valued in ${}^LG(\overline{\mathbb{Q}_\ell})$. It is also conjectured ([BG14, Conjecture 3.15]) that an automorphic representation π is L-algebraic if and only if it is L-arithmetic , i.e. if there is a number field E such that at all unramified primes $\mathfrak{p}$, the Satake parameter of $\pi_\mathfrak{p}$ is defined over E. L-arithmetic representations are particularly conducive to computations as we can in principle perform many computations on them over a fixed number field independent of the primes are interested in.

Much of the existing work on automorphic forms in the context of the Langlands programme focuses on forms whose associated automorphic representations are L-algebraic. One can show that algebraic Hecke characters correspond to compatible families of ℓ-adic Galois characters [Sno09]. The simplest examples of automorphic representations that are not L-algebraic then arise from nonalgebraic Hecke characters. In this paper, we will be interested in arguably the second simplest class of nonalgebraic representations, which arise from Hilbert modular forms of nonparitious weight.

1.1. **Hilbert modular forms.** Hilbert modular forms are a natural generalization of classical modular forms (which are automorphic forms for $\mathrm{GL}_2/\mathbb{Q}$) to totally real fields of higher degree. We refer the reader to Section 2.3 for background on Hilbert modular forms. We write $M_k(\mathfrak{N},\chi)$ for the space of Hilbert modular forms with level $\mathfrak{N}\subset\mathbb{Z}_F$, weight $k\in\mathbb{Z}_{\geq 1}^{[F:\mathbb{Q}]}$ and nebentypus character χ. We say that k is paritious if the entries of k are all congruent modulo 2, and nonparitious otherwise. The theories of Hecke operators and newforms extend to the setting of Hilbert modular forms. The automorphic representation associated to a Hilbert modular newform f is L-algebraic if and only if the weight of f is paritious [BG14].

Hilbert modular forms have q-expansions, and because the space $M_k(\mathfrak{N},\chi)$ is finite-dimensional, we can describe it explicitly by giving the q-expansions (to some precision) of a basis of forms spanning $M_k(\mathfrak{N},\chi)$. As in the setting of modular forms, $M_k(\mathfrak{N},\chi)$ is spanned by Hecke eigenforms. As such, we can access the space by first computing matrices for the action of the Hecke operator $T_\mathfrak{p}$ on $M_k(\mathfrak{N},\chi)$ for sufficiently many $\mathfrak{p}$, and then using these matrices to produce a basis of q-expansions for $M_k(\mathfrak{N},\chi)$. By the Jacquet-Langlands correspondence, we can compute these matrices by studying the Hecke action on certain spaces of quaternionic modular forms for a quaternion algebra B/F. This theory originates with the work of Eichler, and was generalized to the setting of totally real fields by Pizer [PIZ76]. Algorithms for producing these matrices when F has narrow class number 1, $k=(2,\dots,2)$ (parallel weight two), and $\chi=1$ (trivial nebentypus) were invented and implemented by Greenberg-Voight [GV11] for indefinite B and by Dembélé [Dem07] for definite B. These methods were extended to the fields of arbitrary narrow class number and general paritious weights by Voight [Voi10] and Dembélé-Donnelly [DD08] respectively. Given matrices for the Hecke action on $M_k(\mathfrak{N}):=M_k(\mathfrak{N},1)$ for $k=(2,\dots,2)$, Donnelly and Voight [DV21] describe an algorithm for producing a basis of $M_2(\mathfrak{N})$ which extends without difficulty to the case of paritious weight $M_k(\mathfrak{N})$. Their algorithm uses several properties of the finite $\mathbb{Q}$-algebra – the Hecke algebra – generated by the operators $\{T_\mathfrak{p}\}$ acting on $M_k(\mathfrak{N})$. The main difference in this case is that the base field of the Hecke algebra is a subfield of F^{gal} that is only $\mathbb{Q}$ when the weight is parallel. In [Mud26], we extend extend all of the above algorithms to deal with an arbitrary nebentypus character χ, but still under the hypothesis of paritious weight (this work is summarized in Section 4. In the definite case, the extension to general nebentypus was described (but to this author's knowledge not implemented) by Dembélé [Dem07].

In this work, we describe and implement an algorithm for computing Hilbert modular forms of nonparitious weight over fields with narrow class number 1.

Theorem 1.1. *Let F be a totally real field of narrow class number 1. Given an integral ideal $\mathfrak{N}$ of F, a weight $k\in\mathbb{Z}_{\geq 1}^{[F:\mathbb{Q}]}$, and a finite order Hecke character χ of F, there is an algorithm that computes a list of q-expansions (to any given precision) of forms spanning $M_k(\mathfrak{N},\chi)$.*

1.2. **"Elemental" Hecke operators.** We say that a set of linear operators $\mathcal{T}:=\{T\}$ acting on a vector space $V/\mathbb{C}$ is defined over a subfield $E\subset\mathbb{C}$ if there exists an E-basis of V that is preserved by $\mathcal{T}$, i.e. such that in this basis every $T\in\mathcal{T}$ has entries in E. The usual Hecke operators $\{T_\mathfrak{p}\}$ are poorly suited to the nonparitious setting, as there is no number field over which they are all defined [DLP19]. It follows that the Hecke eigenvalues of a normalized eigenform are not defined over a fixed number field. The Hecke eigenvalues of a newform determine the Satake parameters of the associated automorphic representation, so this is exactly the failure of L-arithmeticity in this setting. In particular, the methods of [DV21], which crucially use the fact that the Hecke algebra on the new subspace can be written as a product of number fields, do not work here. The main idea in the proof of Theorem 1.1 is the introduction of "elemental" Hecke operators and corresponding "elemental" Hecke algebra. Given a totally positive generator π of $\mathfrak{p}$ (we are assuming that F has narrow class number 1), the elemental Hecke operator T_π is a rescaling of $T_\mathfrak{p}$ that depends on the choice of π but can be defined over a number field independent of $\mathfrak{p}$. We will show that matrices for the action of the elemental Hecke operators on spaces of quaternionic modular forms can be efficiently computed.

Theorem 1.2. *Let $F, \mathfrak{N}, k,$ and χ be as in Theorem 1.1. (In particular, we continue to assume that F has narrow class number 1.) Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_F$.*

1. *If k is paritious, there is an algorithm for computing the matrix of the $T_\mathfrak{p}$-action on $S_k(\mathfrak{N}, \chi)$.*
2. *If k is nonparitious, and π is a totally positive generator of $\mathfrak{p}$, there is an algorithm for computing the matrix of the T_π-action on $S_k(\mathfrak{N}, \chi)$.*

We then show that replacing Hecke operators with elemental Hecke operators and the usual Hecke algebra with the elemental Hecke algebra allows us to repair the existing algorithms and compute bases as in Theorem 1.1.

In the process of proving Theorem 1.2 and Theorem 1.1, we give alternative proofs of several standard facts (Theorem 5.7, Proposition 5.8, Proposition 5.9) about Hilbert modular forms appearing in e.g. [Shi78]. While these proofs will be of no surprise to the experts, we hope that having self-contained proofs of these results that extend to the nonparitious case will be of some use.

1.3. Previous computations of nonparitious Hilbert modular forms.

We are not the first to compute nonparitious Hilbert modular forms. Buzzard [Buz12] computes the Satake parameters of an explicit CM nonparitious Hilbert modular form of weight $(1, 2)$ via automorphic induction. More recently, Dembélé, Loeffler, and Pacetti [DLP19] associate Galois representations to nonparitious Hilbert modular forms and compute some examples of nonparitious forms. To produce examples, they use the definite method of [Dem07] to compute, for $F = \mathbb{Q}(\sqrt{2})$ and $F = \mathbb{Q}(\sqrt{5})$, the action of the Hecke operators $\{T_\mathfrak{p}\}$ on $M_k(\mathfrak{N}, \chi)$. They also use "naive" Hecke operators which are equivalent to our elemental Hecke operators $\{T_\pi\}$, and compute the naive Hecke eigenvalues of a particular nonparitious newform of weight $(4, 3)$.

In our view, the main conceptual difference between this work and the computations of [DLP19] is that we work entirely with the elemental Hecke operators from the beginning, rather than computing the usual Hecke operators and then rescaling. This may seem like an unimportant distinction, but this point of view lets us bypass issues of square roots and work with a number field independent of $\mathfrak{p}$ (Theorem 5.7) throughout the computation. Furthermore, this allows us to adapt the methods of [DV21] to produce bases of q-expansions for our spaces.

In practice, when computing spaces in paritious weight at multiple levels, we use various tricks – most of which boil down to some version of Shapiro's lemma –

to facilitate efficient computation [DV13]. Our implementation extends these approaches to nonparitious weight in both the definite and indefinite settings, and is integrated with existing machinery for computing tables of Hilbert modular forms. While in some sense these are "implementation details" rather than theoretical differences, these optimizations allow us to efficiently compute spaces of nonparitious forms over higher degree fields and with large levels and weights. We believe that one of the features of the present work is that it is not an ad hoc implementation, but rather part of the robust package for computing with Hilbert modular forms developed in [ABB$^+$26].

1.4. **Organization.** In Section 2, we discuss our notation and review the theory of Hecke characters and Hilbert modular forms. In Section 3, we describe how to produce matrices for the Hecke action on the spaces $S_k(\mathfrak{N}, \chi)$ – this section summarizes new algorithms for computing Hilbert modular forms with nontrivial nebentypus and paritious weight which are discussed in more detail and generality in [Mud26]. In Section 4, we describe in detail the approach of [DV21] to compute q-expansion bases for spaces of Hilbert modular forms of paritious weight. In Section 5, we introduce the elemental Hecke algebra and use it to extend the methods of Section 3 and Section 4 to the nonparitious setting. In Section 6, we give some examples computed using our implementation.

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2. PRELIMINARIES

2.1. Symbols.

- $[n]$ – $\{1, \ldots, n\}$;
- F – totally real field with narrow class number 1;
- $\mathbb{Z}_F$ – ring of integers of F;
- $\mathbb{Z}_\mathfrak{p}$ – completion of $\mathbb{Z}_F$ at a prime ideal $\mathfrak{p} \subset \mathbb{Z}_F$;
- F^{gal} – smallest extension of F that is Galois over $\mathbb{Q}$;
- $F_{>0}$ – totally positive elements of F;
- $I_{>0}$ – totally positive elements of an ideal I;
- $\mathbb{Z}_{F,>0}^\times$ – totally positive units of F;
- $\mathfrak{d}_F$ – different of F;
- $\mathrm{GL}_2^+(F)$ – elements of $\mathrm{GL}_2(F)$ with totally positive determinant;
- $\mathcal{H}$ – complex upper half-plane;
- B – quaternion algebra with center F;
- $B_\infty^\times$ – $\prod'_{v\mid\infty} B_v^\times$;
- $\widehat{B}^\times$ – $\prod_{v\nmid\infty} B_v^\times$;
- $\mathcal{O}$ – (Eichler) order in B;
- $\widehat{\mathcal{O}}^\times$ – $\prod_{v\nmid\infty} \mathcal{O}_v^\times$.

2.2. **Embeddings and multi-index notation.** We fix once and for all an embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. In particular, this restricts to an embedding $\iota\colon F^{\mathrm{gal}} \hookrightarrow \mathbb{R}$. We also fix an ordering $(\sigma_i)_{i\in[n]}$ of the n embeddings of F into F^{gal}. Given an element $x \in F$, we write $x_i := \sigma_i(x) \in F^{\mathrm{gal}}$. Similarly, for a matrix $\gamma \in M_2(F)$, we write $\gamma_i \in M_2(F^{\mathrm{gal}})$ for the matrix obtained by applying σ_i entrywise.

Given a totally positive element $x \in F^{\mathrm{gal}}$ we define $x^{1/s}$ to be the unique s^{th} root y of x in $\overline{\mathbb{Q}}$ such that $\iota(y) \in \mathbb{R}_{>0}$. We can extend this to define $x^{r/s}$ for any totally positive $x \in F^{\mathrm{gal}}$ and $\frac{r}{s} \in \mathbb{Q}$. We will make frequent use of multi-index notation. For $t \in \mathbb{Q}^n$ and $x \in F^{\mathrm{gal}}$, we write $x^t := \prod_i x_i^{t_i} = \prod_i \sigma_i(x)^{t_i} \in \overline{\mathbb{Q}}$. Similarly, for $z \in \mathbb{C}^n$ and $t \in \mathbb{Z}^n$, we write $z^k := \prod_i z_i^{t_i}$. For $t \in \mathbb{Q}$ and $x \in F^{\mathrm{gal}}$, we write $x^{\underline{t}} := \prod_i x_i^t$.

2.3. **Hilbert modular forms.** Let F be a totally real number field of degree $n > 1$ of narrow class number 1. For $z \in \mathcal{H}^n$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F)$, we define

$$\gamma z := \left(\frac{\iota(a_i)z_i + \iota(b_i)}{\iota(c_i)z_i + \iota(d_i)}\right)_{i\in[n]} \in \mathcal{H}^n \qquad \text{and} \qquad j(\gamma, z) := (\iota(c_i)z_i + \iota(d_i))_{i\in[n]} \in \mathbb{C}^n.$$

Given a function $f\colon \mathcal{H}^n \to \mathbb{C}$, $k \in \mathbb{Z}^n_{>0}$, and $\gamma \in \mathrm{GL}_2^+(F)$, we can define another function

$$(f|_k\gamma)(z) := \frac{(\det\gamma)^{k/2}}{j(\gamma, z)^k} f(\gamma z).$$

Let $\mathfrak{N} \subset \mathbb{Z}_F$ be an ideal, and set

$$\mathcal{O}_0(\mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_F)\colon c \in \mathfrak{N} \right\} \qquad \text{and} \qquad \Gamma_0(\mathfrak{N}) := \mathcal{O}_0(\mathfrak{N}) \cap \mathrm{GL}_2^+(F).$$

For a finite order character χ of modulus $\mathfrak{N}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O}_0(\mathfrak{N})$, we define $\chi_0(\gamma) := \chi_0(d)$ – here, following [ABB+26, Section 2.2], χ_0 refers to the restriction of χ to $\prod_{\mathfrak{p}\subset\mathbb{Z}_F} \mathbb{Z}_{F_\mathfrak{p}}^\times$.

Definition 2.1. Let $\mathfrak{N}$ and χ be as above, and fix $k \in \mathbb{Z}^n_{\geq 1}$. A Hilbert modular form of weight k, level $\mathfrak{N}$, and nebentypus χ is a holomorphic function $f\colon \mathcal{H}^n \to \mathbb{C}$ such that for any $\gamma \in \Gamma_0(\mathfrak{N})$, $f|_k\gamma(z) = \chi_0(\gamma)f(z)$. We write $M_k(\mathfrak{N}, \chi)$ for the complex vector space of Hilbert modular forms of level $\mathfrak{N}$ and nebentypus χ.

Remark. It may seem strange that the condition on f depends only on χ_0 and not on all of χ. However, as noted in [ABB+26, Section 2.2], for F with (narrow) class number 1, χ_0 determines χ and there is no distinction. For a discussion of the general case, see [Mud26].

A weight k is parallel if the entries of k are all the same and paritious if the entries of k are all congruent modulo 2. Given a weight k, we write k_i for the i^{th} component of k and $k_0 := \max_i k_i$. The space $M_k(\mathfrak{N}, \chi)$ is a direct sum of the subspace of cusp forms $S_k(\mathfrak{N}, \chi)$ and the subspace of Eisenstein series, as defined in [Shi78]. There are explicit formulas for the Eisenstein series in $M_k(\mathfrak{N}, \chi)$ (see [DK21]), so the problem of computing a basis for $M_k(\mathfrak{N}, \chi)$ essentially reduces to computing a basis for $S_k(\mathfrak{N}, \chi)$. When k is nonparallel there are no Eisenstein series at all, and $M_k(\mathfrak{N}, \chi) = S_k(\mathfrak{N}, \chi)$.

Any $f \in M_k(\mathfrak{N}, \chi)$ has a Fourier expansion

$$f(z) := a_0 + \sum_{\nu \in \mathfrak{d}_{F,>0}^{-1}} a_\nu(f) \exp\left(2\pi i \sum_j \iota(\nu_j) z_j\right)$$

which uniquely determines f.

We call $a_\nu := a_\nu(f)$ the Fourier coefficient of f at ν. Applying the condition in Definition 2.1 to the matrices $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$ for $\epsilon \in \mathbb{Z}_{F,>0}^\times$, we find

$$a_{\epsilon\nu} = \epsilon^{k/2} a_\nu \text{ for all } \epsilon \in \mathbb{Z}_{F,>0}^\times \tag{1}$$

Similarly, applying it to $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ for $\epsilon \in \mathbb{Z}_F^\times$,

$$\chi(\epsilon) = \operatorname{sign}(\epsilon)^k := \prod_i \operatorname{sign}(\epsilon_i)^{k_i} \text{ for all } \epsilon \in \mathbb{Z}_F^\times. \tag{2}$$

To any $\nu \in \mathfrak{d}_{F,>0}^{-1}$, we associate an integral ideal $\mathfrak{n} := (\nu)\mathfrak{d}_F$. The ideal $\mathfrak{n}$ then has a corresponding "ideal coefficient"

$$a_\mathfrak{n}(f) := a_\nu(f) \nu^{(\underline{k_0} - k)/2}. \tag{3}$$

This is well-defined by Equation (1), as replacing ν by $\epsilon\nu$ for $\epsilon \in \mathbb{Z}_{F,>0}^\times$ does not change the value of $a_\mathfrak{n}(f)$.

Given an ideal $\mathfrak{m} \subset \mathbb{Z}_F$, there is an explicit formula for the Hecke operator $T_\mathfrak{m}$ on $f \in M_k(\mathfrak{N}, \chi)$ in terms of the ideal coefficients [Shi78]:

$$a_\mathfrak{n}(T_\mathfrak{m} f) = \sum_{\substack{\mathfrak{m}+\mathfrak{n} \subset \mathfrak{a} \\ \mathfrak{a} \subset \mathbb{Z}_F}} \chi^*(\mathfrak{a}) \operatorname{Nm}(\mathfrak{a})^{k_0 - 1} a_{\mathfrak{n}\mathfrak{m}\mathfrak{a}^{-2}}(f). \tag{4}$$

The Hecke operators satisfy the following relations.

$$T_{\mathfrak{p}^t} = T_\mathfrak{p} T_{\mathfrak{p}^{t-1}} - \operatorname{Nm}(\mathfrak{p})^{k_0 - 1} \chi^*(\mathfrak{p}) T_{\mathfrak{p}^{t-2}} \qquad \text{and} \qquad T_{\mathfrak{n}\mathfrak{m}} = T_\mathfrak{n} T_\mathfrak{m} \text{ if } (\mathfrak{n}, \mathfrak{m}) = 1. \tag{5}$$

Let $F^{\mathrm{gal}}(\chi)$ denote the compositum of F^{gal} and the cyclotomic field in which χ_0 is valued. We can think of the components of k as being indexed by the real embeddings of F. Then, $\operatorname{Gal}(F^{\mathrm{gal}}/\mathbb{Q})$ acts on k by permuting the components, and we write $F' \subset F^{\mathrm{gal}}$ for the field fixed by automorphisms of F^{gal} that preserve k under the aforementioned action ([Shi78, Proposition 1.4]).

We say that a set of linear operators $\{T\}$ acting on a finite-dimensional complex vector space V can be defined over a field K if there is a choice of basis of V such that in this basis, the matrix of every operator in $\{T\}$ has entries in K. The following theorem underpins the algorithms used to produce q-expansions from Hecke matrices in the paritious case.

Theorem 2.2 (Implicit in [Shi78])**.** *The Hecke operators $\{T_\mathfrak{p}\}$ acting on $S_k(\mathfrak{N}, \chi)$ can be defined over $F'(\chi)$ when k is paritious.*

Remark. In many practical settings – for example when F is a quadratic or Galois cubic field and χ is trival or quadratic – we have $F'(\chi) = F$. One can also check that $F'(\chi)$ is the smallest possible coefficient field of a Hilbert modular form in $S_k(\mathfrak{N}, \chi)$ can be defined, as the coefficient field will always contain the nebentypus field and by Equation (1) must contain $\epsilon^{k/2}$ for any $\epsilon \in \mathbb{Z}_F^\times$.

For any $\mathfrak{M} \subset \mathbb{Z}_F$ and $\mathfrak{D} \subset \mathbb{Z}_F$, there is a degeneracy map

$$\iota_{\mathfrak{D}} \colon M_k(\mathfrak{M}, \chi) \hookrightarrow M_k(\mathfrak{M}\mathfrak{D}, \chi) \tag{6}$$

given by

$$a_{\mathfrak{n}}(\iota_{\mathfrak{D}}(f)) := \begin{cases} a_{\mathfrak{n}\mathfrak{D}^{-1}}(f) & \mathfrak{D} | \mathfrak{n} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We write $S_k(\mathfrak{N}, \chi)^{\text{new}}$ to denote the orthogonal complement in $S_k(\mathfrak{N}, \chi)$ (with respect to the Petersson inner product) of the sum of the images of the degeneracy maps $\iota_{\mathfrak{D}}$ over all $\mathfrak{D} | \mathfrak{N}$ such that $\operatorname{cond}(\chi) | \mathfrak{N}\mathfrak{D}^{-1}$. The following is a Hilbert modular forms analogue of Atkin-Lehner-Li theory for modular forms, and is proved in the same way.

Theorem 2.3. *There is a decomposition*

$$M_k(\mathfrak{N}, \chi) \cong \bigoplus_{\substack{\mathfrak{M} | \mathfrak{N} \\ \operatorname{cond}(\chi) | \mathfrak{M}}} \bigoplus_{\mathfrak{D} | \mathfrak{N}\mathfrak{M}^{-1}} \iota_{\mathfrak{D}}(M_k(\mathfrak{M}, \chi)^{\text{new}}).$$

When k is paritious, we define the Hecke algebra $\mathbb{T} := \mathbb{T}_{F'(\chi)}(\mathfrak{N}, k, \chi)$ to be the commutative $F'(\chi)$-algebra generated by the Hecke operators $\{T_{\mathfrak{m}}\}_{\mathfrak{m} \subset \mathbb{Z}_F}$ acting on $S_k(\mathfrak{N}, \chi)$. The "anemic" Hecke algebra $\mathbb{T}_0 := (\mathbb{T}_{F'(\chi)})_0$ is defined similarly but is generated by the Hecke operators $\{T_{\mathfrak{m}} \colon (\mathfrak{m}, \mathfrak{N}) = 1\}$. The decomposition of Theorem 2.3 is a decomposition of $\mathbb{T}_0$-modules.

By the same argument as is used to prove Theorems 5.5.3 and 5.8.2 of [DS05], one can check that $\mathbb{T}_0$ acts semisimply on $S_k(\mathfrak{N}, \chi)$ and that $\mathbb{T}$ acts semisimply on $S_k(\mathfrak{N}, \chi)^{\text{new}}$. Therefore, $S_k(\mathfrak{N}, \chi)^{\text{new}}$ has a basis of $\mathbb{T}$-eigenforms. These eigenforms are called newforms. We say that a $\mathbb{T}$-eigenform f is normalized if the ideal coefficient $a_{(1)}(f)$ is 1. By Equation (4), the coefficient $a_{\mathfrak{p}}(f)$ of a normalized eigenform f is exactly the eigenvalue of $T_{\mathfrak{p}}$ on f.

Proposition 2.4 ([Shi78, Proposition 2.8])**.** *Let $f \in S_k(\mathfrak{N}, \chi)$ be a normalized eigenform with k paritious. Then, $\mathbb{Q}(\{a_{\mathfrak{p}}(f)\})$ is a finite extension of $\mathbb{Q}$.*

Proposition 2.5 (Special case of [Shi78, Proposition 2.6])**.** *If k is paritious, $f \in S_k(\mathfrak{N}, \chi)$, and τ is an automorphism of $\mathbb{C}$ fixing $F'(\chi)$, then*

$${}^{\tau}f(z) := \sum_{\nu \in \mathfrak{d}_{F,>0}^{-1}} \tau(a_{\nu}(f)) \exp\left(2\pi i \sum_j \iota(\nu_j) z_j\right)$$

is an element of $S_k(\mathfrak{N}, \chi)$. Furthermore, if f is a $\mathbb{T}_0$ or $\mathbb{T}$-eigenform, then so is ${}^{\tau}f$.

We give proofs of Proposition 2.4 and Proposition 2.5 assuming Theorem 2.2 in Section 4. In Section 5, we extend Theorem 2.2 to the nonparitious case (as Theorem 5.7), and from it deduce generalizations of Proposition 2.4 (as Proposition 5.8), and Proposition 2.5 (as Proposition 5.9).

3. Quaternionic modular forms and computing matrices for $T_{\mathfrak{p}}$

Theorem 3.1. *There is an explicit algorithm, which given a paritious weight $k \in \mathbb{Z}_{\geq 2}^n$, a level $\mathfrak{N} \subset \mathbb{Z}_F$, a finite order nebentypus χ of modulus $\mathfrak{N}$, and a prime $\mathfrak{p}$, returns a matrix for the Hecke operator $T_{\mathfrak{p}}$ on $S_k(\mathfrak{N}, \chi)$ in a basis independent of $\mathfrak{p}$ and over an extension of $F'(\chi)$ independent of $\mathfrak{p}$.*

The full proof of Theorem 3.1 can be found in [Mud26] (where the case of nontrivial narrow class group is also addressed) and proceeds by establishing a Hecke module isomorphism between $S_k(\mathfrak{N}, \chi)$ and an appropriate space of quaternionic modular forms on which Hecke matrices can be computed explicitly. In this section, we outline some features of the construction that will be relevant in Section 5 when we compute spaces of nonparitious forms.

Remark. It may seem surprising that even though Theorem 2.2 indicates that the matrices should be defined over $F'(\chi)$ itself, Theorem 3.1 produces matrices defined over a finite extension of $F'(\chi)$. This is because in practice, it is easier to work over an extension of $F'(\chi)$ splitting B. In Section 4, we will see that Theorem 3.1 is already enough to let us compute a basis of q-expansions over $F'(\chi)$ for $S_k(\mathfrak{N}, \chi)$. Once such a basis has been computed, we can compute matrices for the Hecke action over $F'(\chi)$ using Equation (4) and linear algebra should we desire.

Let B/F be a quaternion algebra with discriminant Δ_B. Let r (resp. s) be the number of infinite places split (resp. ramified) in B. Fix a level $\mathfrak{M} \subset \mathbb{Z}_F$ such that $(\mathfrak{M}, \Delta_B = (1)$ and a finite order Hecke character χ with $\operatorname{cond}(\chi) | \mathfrak{M}$. Let $\mathbb{Z}_{\mathfrak{M}} := \prod_{\mathfrak{p}|\mathfrak{M}} \mathbb{Z}_{\mathfrak{p}}$. Given a splitting $\iota_{\mathfrak{M}} \colon B \hookrightarrow M_2(\prod_{v|\mathfrak{M}} F_v)$, we write $\mathcal{O}_0(\mathfrak{M})$ for the the Eichler order of level $\mathfrak{M}$. To avoid clutter, we write $\mathcal{O} := \mathcal{O}_0(\mathfrak{M})$ when there is no ambiguity. For $\hat{x} \in \widehat{B}^\times$ such that

$$(\hat{x})_{v|\mathfrak{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_{\mathfrak{M}}) \tag{8}$$

we define $\chi_0(\hat{x}) := \chi_0(d)$. Similarly, if the image of $x \in \widehat{B}^\times$ under the diagonal embedding to $\widehat{B}^\times$ is a $\hat{x}$ satisfying Equation (8), we define $\chi_0(x) := \chi_0(\hat{x})$. Choose a splitting

$$B^\times \hookrightarrow \prod_{\substack{v|\infty \\ v \text{ ramified}}} B_v^\times \hookrightarrow \mathrm{GL}_2(\mathbb{C})^s.$$

Precomposing with this splitting, the right $\mathrm{GL}_2(\mathbb{C})^s$-representation

$$W_k(\mathbb{C}) := \bigotimes_{\substack{v|\infty \\ v \text{ ramified}}} \left(\operatorname{Sym}^{k_v - 2} \mathbb{C}^2 \otimes (\det)^{(\underline{k_0} - k_v)/2}\right) \tag{9}$$

gives rise to a representation of $B^\times$ over $\mathbb{C}$. In practice, we actually pick a number field K containing F^{gal} and splitting B, and do all our computation over the compositum $K(\chi)$ containing K and the field of values of χ.

Recall the definition of a quaternionic modular form with nebentypus from [Mud26, Definition 3.1]. We denote the subspace of quaternionic cusp forms of level $\mathfrak{M}$ and nebentypus χ by $S_k^B(\mathfrak{M}, \chi)$. When working with quaternionic modular forms, it will be helpful to understand the double coset space $B_+^\times \backslash \widehat{B}^\times / \mathcal{O}^\times$. Let H be the cardinality of this double coset space, and pick representatives $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_H\} \subset \widehat{B}^\times$. Given $\phi \in M_k^B(\mathfrak{M}, \chi)$ and $h \in [H]$, we can define a function

$$\begin{aligned} \phi_h \colon \mathcal{H}^s &\longrightarrow W_k(\mathbb{C}) \\ z &\longmapsto f(z, \hat{\alpha}_h). \end{aligned}$$

For any function $\phi_h \colon \mathcal{H}^r \to W_k(\mathbb{C})$, we define an action

$$\phi_h|_k\gamma := \frac{(\det\gamma)^{k/2}}{j(\gamma, z)^k}\phi_h(\gamma z)^{\gamma}. \tag{10}$$

Let $\mathcal{O}_h := \hat{\alpha}_h\hat{\mathcal{O}}\hat{\alpha}_h^{-1} \cap B$, and define

$$M_k^B(\mathfrak{M}, \chi; h) := \ \{\phi_h \colon \mathcal{H}^r \longrightarrow W_k(\mathbb{C}) \colon \ \phi_h|_k\gamma = \chi_0(\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h)\phi_h \ \text{ for } \gamma \in \mathcal{O}_h^{\times}\}.$$

Note that $\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h \in \widehat{\mathcal{O}}^{\times}$, so it makes sense to evaluate χ_0 on it.

Lemma 3.2. *The map*

$$\begin{array}{rcl} \Phi\colon\ M_k^B(\mathfrak{M}, \chi) & \longrightarrow & \bigoplus_{h=1}^{H} M_k^B(\mathfrak{M}, \chi; h) \\ \phi & \longmapsto & (\phi_h)_{h\in[H]} \end{array}$$

is an isomorphism.

Proof. This is [Mud26, Lemma 4.2]. □

We now define Hecke operators on $M_k^B(\mathfrak{N}, \chi)$. Given a prime ideal $\mathfrak{p} \subset \mathbb{Z}_F$, let $\hat{\pi} \in \widehat{B}^{\times}$ be an element which at places $v \neq \mathfrak{p}$ is 1 and at $v = \mathfrak{p}$ is an element whose reduced norm is a uniformizer for $F_{\mathfrak{p}}$. The choice of $\hat{\pi}$ does not affect the double coset $\widehat{\mathcal{O}}^{\times}\hat{\pi}\widehat{\mathcal{O}}^{\times}$, which is all that will matter. Equivalently, we may choose $\hat{\pi}$ such that $\operatorname{nrd}(\hat{\pi})\hat{\mathbb{Z}}_F \cap F = \mathfrak{p}$. Let P be $\operatorname{Nm}(\mathfrak{p}) + 1$ if $\mathfrak{p} \nmid \mathfrak{N}$ and $\operatorname{Nm}(\mathfrak{p})$ otherwise. There exist elements $\{\hat{\pi}_j\}_{j=1}^{P} \subset \widehat{B}^{\times}$ such that

$$\widehat{\mathcal{O}}^{\times}\backslash\widehat{\mathcal{O}}^{\times}\hat{\pi}\widehat{\mathcal{O}}^{\times} \ = \bigsqcup_j \hat{\mathcal{O}}^{\times}\hat{\pi}_j.$$

For $\phi \in M_k^B(\mathfrak{M}, \chi)$, we define the Hecke operator

$$(T_{\mathfrak{p}}\phi)(z, \hat{x}) := \sum_j \phi(z, \hat{x}\hat{\pi}_j^{-1})\chi_0(\hat{\pi}_j). \tag{11}$$

The right-hand side of Equation (11) is independent of the choices of $\hat{\pi}_j$. We can ask how the Hecke operator interacts with the isomorphism of Lemma 3.2.

Lemma 3.3. *There exist:*

1. *A function $j^* \colon [H] \to [H]$ for every $j \in [P]$;*
2. *Elements*

$$\left\{\varpi_{j,h} \in \hat{\alpha}_{j^*(h)}\hat{\mathcal{O}}^{\times}\hat{\pi}_j\hat{\alpha}_h^{-1} \ \cap B_+^{\times}\right\}_{\substack{j\in[P]\\ h\in[H]}},$$

where $\varpi_{j,h}$ is well-defined up to multiplication on the left by $\mathcal{O}_{j^*(h)}^{\times}$;

such that

$$(T_{\mathfrak{p}}\phi)_h(z) = \sum_{j=1}^{P} \chi_0(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)(\phi_{j^*(h)}|_k\varpi_{j,h})(z). \tag{12}$$

Proof. This is [Mud26, Lemma 4.3]. □

We discuss effectively computing Lemma 3.3 in [Mud26]. For our purposes, the main point is that, once we have chosen representatives $\{\hat{\alpha}_h\}_{h\in[H]}$, there is a formula for $T_{\mathfrak{p}}$ depending only on a set of elements $\{\varpi_{j,h}\}_{\substack{j\in[P]\\ h\in[H]}}$, each of which can be chosen up to multiplication on the left by an element of $\mathcal{O}_{j^*(h)}^{\times}$.

The reason we care about the Hecke action on the spaces $S_k^B(\mathfrak{N},\chi)$ is the (Eichler-Shimizu-)Jacquet-Langlands correspondence (see e.g.[Mud26, Theorem 2.3]). When n (the degree of F) is odd, we choose B to be an indefinite quaternion algebra ramified at all but one of the infinite places. When n is even, we choose B to be a definite quaternion algebra. In either case, we may take B to be unramified at all finite places, so $\Delta_B = (1)$. For such a B, $S_k^B(\mathfrak{N},\chi) \cong S_k(\mathfrak{N},\chi)$ on the nose by (Eichler-Shimizu-)Jacquet-Langlands. In this case, the matrices for the Hecke action on $S_k^B(\mathfrak{N},\chi)$ are exactly the matrices for the Hecke action on $S_k(\mathfrak{N},\chi)$.

Let

$$V_k(\mathbb{C}) := \bigotimes_{v|\infty}\left(\mathrm{Sym}^{k_v-2}\,\mathbb{C}^2 \otimes (\det)^{(\underline{k_0}-k_v)/2}\right). \tag{13}$$

When B is definite, $V_k(\mathbb{C}) \cong W_k(\mathbb{C})$, but when B is indefinite, $V_k(\mathbb{C})$ includes factors from the split places that $W_k(\mathbb{C})$ does not. Let $\rho\colon B^{\times} \to \mathrm{End}(V_k(\mathbb{C}))$ be the representation associated to $V_k(\mathbb{C})$ (Equation (13)). If B is definite, Lemma 3.3 shows that the Hecke operator $T_{\mathfrak{p}}$ on $S_k^B(\mathfrak{N},\chi)$ acts by an $[H]\times[H]$ block matrix where each block is a linear combination (with coefficients in $\mathbb{Q}(\chi)$) of matrices $\rho(\varpi)$ for various $\varpi \in \{\varpi_{j,h}\}_{\substack{j\in[P]\\ h\in[H]}}$. If B is indefinite, the situation is more complicated, but applying a version of the Eichler-Shimura isomorphism ([Mud26, Theorem 5.2]) to compute Equation (12), we end up with a block matrix where each block consists of a linear combination (with coefficients in $\mathbb{Q}(\chi)$) of products $\rho(\gamma)\rho(\varpi_{j,h})$, for $\gamma \in B^1$. We can realize the representation ρ explicitly by working over a number field $K/F'(\chi)$ splitting B. Putting everything together, Theorem 3.1 is proved.

4. COMPUTING SPACES OF HILBERT MODULAR FORMS OF PARITIOUS WEIGHT

Our goal in this section is to compute a basis of $S_k(\mathfrak{N},\chi)$ (i.e. produce explicit q-expansions of forms spanning the space). Following the approach of [DV21], we proceed as follows:

1. Use Theorem 3.1 to compute "full" Hecke matrices (as opposed to matrices restricted to the new subspace) for the Hecke action on $S_k(\mathfrak{M},\chi)$ for $\mathfrak{M} \subset \mathbb{Z}_F$ such that $\mathrm{cond}(\chi)|\mathfrak{M}$ and $\mathfrak{M}|\mathfrak{N}$.
2. Use these matrices and properties of the Hecke algebra $\mathbb{T}$ to produce a subspace $V_f \subset S_k(\mathfrak{M},\chi)$ for each Galois ($\mathrm{Aut}_{F'(\chi)}(\mathbb{C})$) conjugacy class of newforms.
3. Use Equation (5) to compute matrices for $\{T_{\mathfrak{m}}\}$, where $\mathfrak{m} \subset \mathbb{Z}_F$ ranges over all ideals, from the matrices $\{T_{\mathfrak{p}}\}$.
4. Use properties of $\mathbb{T}$ to compute a basis of q-expansions with coefficients in $F'(\chi)$ spanning each V_f.
5. Use Theorem 2.3 and Equation (7) to assemble a basis for $S_k(\mathfrak{N},\chi)$ over $F'(\chi)$.

A key point is that throughout, we work over a number field independent of the prime $\mathfrak{p}$.

Remark. In any given application of the algorithm, we want to compute the q-expansions of a basis up to some precision, and as such only compute finitely many Hecke operators $\{T_\mathfrak{p}\}$.

4.1. An algorithm for computing $S_k(\mathfrak{N},\chi)$ for k paritious.

4.1.1. *Compute "full" Hecke matrices.* Applying Theorem 3.1 to every $\mathfrak{M}|\mathfrak{N}$ such that $\operatorname{cond}(\chi)|\mathfrak{M}$, we can compute matrices for the action of $T_\mathfrak{p}$ on $S_k(\mathfrak{M},\chi)$ for any $\mathfrak{M}$ and any $\mathfrak{p}$.

4.1.2. *Restrict the Hecke matrices to Galois orbits of newforms.* We first want to use the full Hecke matrices to identify the subspace $S_k(\mathfrak{N},\chi)^{\text{new}} \subset S_k(\mathfrak{N},\chi)$. Let $\mathfrak{M}|\mathfrak{N}$ be such that $\operatorname{cond}(\chi)|\mathfrak{M}$. For a level $\mathfrak{M} \subset \mathbb{Z}_F$ and a Hecke operator $T_\mathfrak{p}$ for $\mathfrak{p} \nmid \mathfrak{N}$, let $\mu_{\mathfrak{M},\mathfrak{p}}$ be the squarefree part of the characteristic polynomial of $T_\mathfrak{p}$ acting on $S_k(\mathfrak{M},\chi)$. Then, consider the operator $\operatorname{im}\mu_{\mathfrak{M},\mathfrak{p}}(T_\mathfrak{p})$, where we apply the characteristic polynomial of the $T_\mathfrak{p}$-action on $S_k(\mathfrak{M},\chi)$ to the Hecke matrix of the $T_\mathfrak{p}$-action on $S_k(\mathfrak{N},\chi)$. Anything in the image of a degeneracy map from level $\mathfrak{M}$ lies in $\ker\mu_{\mathfrak{M},\mathfrak{p}}(T_\mathfrak{p})$, Because $T_\mathfrak{p}$ acts semisimply, so too does $\mu_{\mathfrak{M},\mathfrak{p}}(T_\mathfrak{p})$, so $S_k(\mathfrak{N},\chi) = \ker\mu_{\mathfrak{M},\mathfrak{p}}(T_\mathfrak{p}) \oplus \operatorname{im}\mu_{\mathfrak{M},\mathfrak{p}}(T_\mathfrak{p})$. Because $T_\mathfrak{p}$ is normal on $S_k(\mathfrak{M},\chi)$ with respect to the Petersson inner product, its kernel and image are orthogonal. The subspace of $S_k(\mathfrak{N},\chi)$ "old at $\mathfrak{N}/\mathfrak{M}$" is then $\bigcap_\mathfrak{p}\ker\mu_{\mathfrak{M},\mathfrak{p}}(T_\mathfrak{p})$, so its orthogonal complement is "new at $\mathfrak{N}/\mathfrak{M}$". Since $S_k(\mathfrak{N},\chi)^{\text{new}}$ is the intersection of the subspaces which are new at $\mathfrak{N}/\mathfrak{M}$ for all $\mathfrak{M}|\mathfrak{N}$, the following follows.

Proposition 4.1 ([DV21])**.**

$$S_k(\mathfrak{N},\chi)^{\text{new}} = \bigcap_{\mathfrak{M}} \sum_{\substack{\mathfrak{p}\subset\mathbb{Z}_F\\ \mathfrak{p}\nmid\mathfrak{N}}} \operatorname{im}\mu_{\mathfrak{M},\mathfrak{p}}(T_\mathfrak{p}),$$

where the intersection is over $\mathfrak{M}$ *such that* $\mathfrak{M}|\mathfrak{N}$*,* $\mathfrak{M}\neq\mathfrak{N}$*, and* $\operatorname{cond}(\chi)|\mathfrak{M}$*.*

Using Proposition 4.1, we can determine the new subspace explicitly – if we compute at the smaller levels first, we can compute the dimension of $S_k(\mathfrak{N},\chi)^{\text{new}}$ using Theorem 2.3, and can compute the intersection of Proposition 4.1 at more and more primes until the dimension of the intersection is correct. Therefore, we can restrict our full Hecke matrices to the new subspace.

We now take a brief detour to understand the structure of the full Hecke algebra $\mathbb{T} := \mathbb{T}_{F'(\chi)}$ (here, we are assuming Theorem 2.2). Writing $\mathbb{T}_\mathbb{C} := \mathbb{T}\otimes_{F'(\chi)}\mathbb{C}$, there is a perfect pairing

$$\begin{aligned} \Psi\colon\ \mathbb{T}_\mathbb{C}\times S_k(\mathfrak{N},\chi) &\longrightarrow \mathbb{C}\\ (T,f) &\longmapsto a_1(Tf) \end{aligned}$$

The perfectness of Ψ can be proved in the same way as the analogous result for modular forms (see e.g. [DI95, Proposition 12.4.13]) – in one direction, if for some $f\in S_k(\mathfrak{N},\chi)$ we have $0=\Psi(T_\mathfrak{m},f)=a_1(T_\mathfrak{m}f)=a_\mathfrak{m}(f)$ for all $T_\mathfrak{m}\in\mathbb{T}_\mathbb{C}$, then $f=0$, and in the other direction, if for some $T\in\mathbb{T}_\mathbb{C}$ we have $0=\Psi(T,f)=a_1(Tf)$ for all f ranging through an eigenbasis in $S_k(\mathfrak{N},\chi)$, then T acts by 0 on the whole space. This pairing induces a bijection between $\mathbb{C}$-algebra homomorphisms from $\mathbb{T}_\mathbb{C}$ to $\mathbb{C}$ and and normalized Hecke eigenforms in $S_k(\mathfrak{N},\chi)$.

By Proposition 4.1, the new subspace can be defined over $F'(\chi)$. Because the Hecke operators preserve the new subspace, it follows that the restriction of $\mathbb{T}$ to

$S_k(\mathfrak{N},\chi)^{\mathrm{new}}$ is also an $F'(\chi)$-algebra, which we call $\mathbb{T}^{\mathrm{new}}$. Because $\mathbb{T}$ acts semisimply on the new subspace, we can write $\mathbb{T}^{\mathrm{new}} \cong \prod_{f\in S} K_f$ for field extensions $K_f/F'(\chi)$ and S some finite index set. The pairing Ψ restricts to a pairing between $\mathbb{T}^{\mathrm{new}}_{\mathbb{C}}$ and $S_k(\mathfrak{N},\chi)^{\mathrm{new}}$, and any element of $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{T}^{\mathrm{new}}_{\mathbb{C}},\mathbb{C})$ is an element of $\mathrm{Hom}_{F'(\chi)}(K_f,\mathbb{C})$ for some $f\in S$. We deduce that S can be taken to be a set of representatives of Galois orbits of newforms under $\mathrm{Aut}_{F'(\chi)}(\mathbb{C})$. In particular, the coefficient field of every normalized newform is the image of some K_f under an embedding to $\mathbb{C}$, proving Proposition 2.4 for newforms. It also follows that $\mathrm{Aut}_{F'(\chi)}(\mathbb{C})$ acts on $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{T}^{\mathrm{new}}_{\mathbb{C}},\mathbb{C})$ and hence that if f is a newform, so is ${}^{\tau}f$. Since the newforms span $S_k(\mathfrak{N},\chi)^{\mathrm{new}}$, Proposition 2.5 follows for forms in the new subspace. We can now prove Proposition 2.4 and Proposition 2.5 in full by observing that the degeneracy maps in Equation (6) commute with Galois automorphisms and do not change the coefficient field.

We have the decomposition

$$S_k(\mathfrak{N},\chi)^{\mathrm{new}} \cong \bigoplus_{f\in S} V_f, \tag{14}$$

where each V_f is the the ($[K_f : F'(\chi)]$-dimensional) span of a $\mathrm{Aut}_{F'(\chi)}(\mathbb{C})$-conjugacy class of the newform $f\in S$.

By the Chinese remainder theorem, $\mathbb{T}\cong\prod_f K_f$ is generated by a single element $T\in\mathbb{T}$ (not necessarily a $T_{\mathfrak{p}}$, but some element nonetheless). Because $\mathbb{T}$ acts semisimply on $S_k(\mathfrak{N},\chi)^{\mathrm{new}}$, we have the decomposition

$$S_k(\mathfrak{N},\chi)^{\mathrm{new}} \cong \bigoplus_{\mu'\mid\mu_{\mathfrak{N},T}} \ker\mu',$$

where the sum ranges over the factors μ' of $\mu(\mathfrak{N},T)$ over $F'(\chi)$. Since T is a generator for $\mathbb{T}$, the factors μ' are in bijection with the factors K_f of $\mathbb{T}$ and the subspaces $\ker\mu'$ are in bijection with the subspaces V_f.

4.1.3. *Produce matrices of $\{T_{\mathfrak{m}}\}$ from the matrices of $\{T_{\mathfrak{p}}\}$.* For each V_f, we can apply the identities in Equation (5) to compute $T_{\mathfrak{m}}|_{V_f}$ for any ideal $\mathfrak{m}\subset\mathbb{Z}_F$. In practice, we want to compute $\{T_{\mathfrak{m}}|_{V_f} : \mathfrak{m}\subset\mathbb{Z}_F, \mathrm{Nm}(\mathfrak{m})\le X\}$ for some bound X. Using the identities in Equation (5), and processing the ideals $\mathfrak{m}$ in order of the number of prime factors of $\mathfrak{m}$ (with multiplicity), we can use dynamic programming to compute $\{T_{\mathfrak{m}}|_{V_f}\}$ with one additional matrix multiplication per ideal $\mathfrak{m}$.

4.1.4. *Compute q-expansions of an $F'(\chi)$-basis of each newform orbit.* Each V_f is a simple $\mathbb{T}$-module on which $\mathbb{T}$ acts as a field extension $K_f/F'(\chi)$. This K_f is exactly the coefficient field of the newform orbit representative f). Let T be a generator of $\mathbb{T}|_{V_f}$. Letting $d := [K_f : F'(\chi)]$, given any $g\in V_f$, $\{T^j g\}_{j=0}^{d-1}$ is a basis for V_f. We will take $g := \sum_{\tau\in\mathrm{Hom}_{F'(\chi)}(K_f,\mathbb{C})} {}^{\tau}f$, where ${}^{\tau}f$ is as in Proposition 2.5. Then, g has coefficients in $F'(\chi)$. The following lemma lets us compute the ideal coefficients of $\{T^j g\}_{j=0}^{d-1}$.

Lemma 4.2. $a_{\mathfrak{n}}(T^j g) = \mathrm{tr}(T^j T_{\mathfrak{n}})$.

Proof. For a $\mathbb{T}$-eigenform $f\in S_k(\mathfrak{N},\chi)^{\mathrm{new}}$, let $\lambda_T(f)$ denote the eigenvalue of the generator T acting on f. Then,

$$\mathrm{tr}(T^j T_{\mathfrak{n}}) = \sum_{\tau}\lambda_T(f_\tau)^j a_{\mathfrak{n}}({}^{\tau}f) \;= a_{\mathfrak{n}}\left(\sum_{\tau}\lambda_T({}^{\tau}f)^j({}^{\tau}f)\right) \;= a_{\mathfrak{n}}(T^j g).$$

□

Because the coefficients of g are in $F'(\chi)$, we can use Lemma 4.2 to produce q-expansions $\{T^j g\}_{j=0}^{d-1}$ spanning V_f with coefficients in $F'(\chi)$. In particular, we can do this directly from the matrix $T \in \mathbb{T}$, which has entries in $F'(\chi)$. As such, all of our computation can be performed over the field $F'(\chi)$ – we never actually work with the coefficient fields $K_f/F'(\chi)$.

4.1.5. *Assemble the bases of newform orbits to produce a basis for $S_k(\mathfrak{N}, \chi)$.* Repeat the previous three steps to produce the q-expansions of a basis of $S_k(\mathfrak{M}, \chi)$ for all $\mathfrak{M}|\mathfrak{N}$ such that $\mathrm{cond}(\chi)|\mathfrak{M}$. Applying Theorem 2.3 with the degeneracy maps as in Equation (6), we obtain the q-expansions of a basis of $S_k(\mathfrak{N}, \chi)$ with coefficients in $F'(\chi)$.

4.2. **Forms of partial weight one.** We say that a Hilbert modular form has partial weight one if at least one component of its weight is 1. Theorem 3.1 does not let us access Hecke matrices on spaces of Hilbert modular forms of partial weight one. For these, we can apply the Hecke stability method of Schaeffer [Sch15] (see also [MS15, ABB$^+$26] for the extension to Hilbert modular forms). Concretely, choosing an Eisenstein series $E \in M_l(\mathfrak{N}, \psi)$ nonvanishing at the cusp at infinity, $S_k(\mathfrak{N}, \chi)$ is contained in the Hecke stable subspace U of the space of meromorphic modular quotients $V := \frac{S_{k+l}(\mathfrak{N}, \chi\psi)}{E}$. We can efficiently compute V using the fast multiplication and division algorithms in [ABB$^+$26], and can compute U from V using the formula in Equation (4). In the case of classical modular forms, Schaeffer proves that $S_k(\mathfrak{N}, \chi) = U$, i.e. that forms in U are in fact holomorphic. We expect his proof to generalize to the Hilbert modular setting, but do not assume this. Instead, we can compute V as above and intersect V with its Hecke translates until we produce a candidate Hecke stable subspace $\tilde{U}$. We then verify that each element in a basis of $\tilde{U}$ is genuinely holomorphic by checking that its square (which has weight in $\mathbb{Z}_{\geq 2}^n$) lies in $S_{2k}(\mathfrak{N}, \chi^2)$.

5. Computing spaces of Hilbert modular forms of nonparitious weight

5.1. **Fourier coefficients and elemental Hecke operators.** In many settings, the ideal coefficients $\{a_\mathfrak{n}\}$ are the more intrinsic way to think about the coefficients of a Hilbert modular form.

For example, the L-function associated to f, up to twist, is defined on $\mathrm{Re}(s) > 1$ as

$$L(s, f) = \sum_{\mathfrak{n} \subset \mathcal{O}_F} a_\mathfrak{n} \, \mathrm{Nm}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} (1 - a_\mathfrak{p} \, \mathrm{Nm}(\mathfrak{p})^{-s} + \chi^*(\mathfrak{p}) \, \mathrm{Nm}(\mathfrak{p})^{k_0 - 1 - 2s})^{-1}, \tag{15}$$

where χ^* evaluates to 0 if $\mathfrak{p} \mid \mathfrak{N}$.

Many constructions of Hilbert modular forms (e.g. as Eisenstein series, CM forms, base change forms, etc.) are given naturally as formulas for the $\{a_\mathfrak{p}\}$, which can then be converted into a Fourier expansion using Equation (3). The procedure in Section 4 is in this vein – from the Hecke matrices $\{T_\mathfrak{p}\}$, we produce the Hecke matrices $\{T_\mathfrak{n}\}$, from these extract the ideal coefficients $\{a_\mathfrak{n}(f)\}$ for f ranging over a basis of $S_k(\mathfrak{N}, \chi)$, and from these can recover the Fourier coefficients of a basis.

This all works in paritious weight because of Theorem 2.2, which guarantees that the $T_\mathfrak{p}$ are defined over $F'(\chi)$. When the weight is nonparitious however, this story breaks down because Theorem 2.2 and Proposition 2.4 do not hold. The field of definition of the matrix of the Hecke operator $T_{\mathfrak{p}}$ acting on $S_k(\mathfrak{N}, \chi)$ and (relatedly) the smallest field containing the ideal coefficient $a_{\mathfrak{p}}(f)$ for a normalized eigenform $f \in S_k(\mathfrak{N}, \chi)$ both depend on the prime $\mathfrak{p}$ when k is nonparitious – this field will generally contain $\sqrt{\pi}$ for any totally positive generator π of $\mathfrak{p}$. In particular, $\mathbb{Q}(\{a_{\mathfrak{n}}(f)\})$ will be infinite. Because in practice we only want to compute finitely many terms of q-expansions, we could address this issue by working in a very large field containing the fields of definition of the finitely many $\{a_{\mathfrak{p}}\}$ that we want to compute. However, this would be extremely inefficient at high precisions, and would introduce many potential errors when coercing between different number fields. We will take a different approach.

The first observation is that while the ideal coefficients of a nonparitious eigenform are not defined over a number field, the Fourier coefficients will be.

Theorem 5.1 (Proposition 1.3 of [Shi78])**.** *If $f \in M_k(\mathfrak{N}, \chi)$ is a normalized Hecke eigenform, then $\mathbb{Q}(\{a_\nu(f)\})$ is a finite extension of $F'(\chi)$.*

One way to think about this is the square root factors in Equation (3) (which are not generally elements of F^{gal} when k is nonparitious) exactly cancel out the square roots which cause $\mathbb{Q}(\{a_{\mathfrak{n}}(f)\})$ to be infinite in the first place. Motivated by Theorem 5.1, we will define "elemental" Hecke operators, indexed by totally positive elements of F. These elemental Hecke operators act on $S_k(\mathfrak{N}, \chi)$ via matrices defined over $F'(\chi)$ (Theorem 5.7). As such, we may replace the usual Hecke algebra (which only makes sense over $\overline{\mathbb{Q}}$ in general) with the "elemental" Hecke algebra, the $F'(\chi)$-algebra generated by the elemental Hecke operators. This elemental Hecke algebra has all of the nice properties that the usual Hecke algebra has in the paritious case. Intuitively, just as the usual Hecke algebra realizes the ideal coefficients as eigenvalues, the elemental Hecke algebra realizes the Fourier coefficients as eigenvalues. This will let us prove Theorem 1.1, and along the way, provide alternative proofs of Theorem 2.2, Theorem 5.1, Proposition 2.4, and Proposition 2.5.

5.2. **Elemental Hecke operators and the elemental Hecke algebra.**

Definition 5.2. Let μ be a totally positive generator for $\mathfrak{m}$. (Here, we use that $\mathrm{Cl}_F^+ = 1$.) We define the elemental Hecke operator

$$T_\mu := \mu^{(k - \underline{k_0})/2} T_{\mathfrak{m}}, \tag{16}$$

where $T_{\mathfrak{m}}$ is defined as in Equation (4).

Pick a totally positive generator δ for the different $\mathfrak{d}_F$. Let f be a normalized $\mathbb{T}$-eigenform. Combining Definition 5.2 with Equation (3),

$$T_\mu f = \mu^{(k-\underline{k_0})/2} a_{\mathfrak{m}}(f) f = \mu^{(k-\underline{k_0})/2} (\delta^{-1}\mu)^{(\underline{k_0}-k)/2} a_{\delta^{-1}\mu}(f) f = \delta^{(k-\underline{k_0})/2} a_{\delta^{-1}\mu}(f) f.$$

Writing $\tilde{f} := \delta^{(k-\underline{k_0})/2} f$, we see that

$$T_\mu \tilde{f} = a_{\delta^{-1}\mu}(\tilde{f})\tilde{f}.$$

The eigenvalues of T_μ therefore give the coefficient $a_{\delta^{-1}\mu}$ of a scalar multiple of the original normalized eigenform.

Applying Equation (3) to Equation (4), we can produce a formula for the Fourier coefficients of $T_\pi f$ in terms of those of f. Writing $נ, \mu,$ and α for totally positive generators of $\mathfrak{n}, \mathfrak{m}$, and $\mathfrak{a}$,

$$a_{\delta^{-1}\mathfrak{j}}(T_\mu f) = \sum_{\mathfrak{n}+\mathfrak{m}\subset\mathfrak{a}} \chi^*(\mathfrak{a})\,\mathrm{Nm}(\mathfrak{a})^{k_0-1}\alpha^{k-\underline{k_0}}\, a_{\delta^{-1}\mathfrak{j}\mu\alpha^{-2}}(f). \tag{17}$$

Lemma 5.3. *The elemental Hecke operators $\{T_\pi\}$ on $S_k(\mathfrak{N},\chi)$ are defined over $F'(\chi)$ for any $k \in \mathbb{Z}^n_{\geq 2}$.*

Proof. By Jacquet-Langlands, it suffices to show this result for the Hecke matrices on $S^B_k(\mathfrak{N},\chi)$ for some B such that $\Delta_B = (1)$. Let $\rho\colon B^\times \to \mathrm{End}(V_k(\mathbb{C}))$ be the representation associated to $V_k(\mathbb{C})$ (Equation (13)). If B is definite, then one can show from Lemma 3.3 that the Hecke operator $T_\mathfrak{p}$ on $S^B_k(\mathfrak{N},\chi)$ acts by an $[H]\times[H]$ block matrix where each block is a linear combination (with coefficients in $\mathbb{Q}(\chi)$) of matrices $\rho(\varpi)$ for various $\varpi \in \{\varpi_{j,h}\}_{\substack{j\in[P]\\ h\in[H]}}$. If B is indefinite, then the situation is more complicated, but again we end up with a block matrix where each block consists of a linear combination (with coefficients in $\mathbb{Q}(\chi)$) of products $\rho(\gamma)\rho(\varpi_{j,h})$, for $\gamma \in B^1$. As such, the matrices for the action of $T_\mathfrak{p}$ on $M^B_k(\mathfrak{N},\chi)$ can be defined over a field containing the field of definitions of χ, $\rho(\gamma)$ for $\gamma\in B^1$, and $\{\rho(\varpi_{j,h})\}_{j,h}$.

When k is paritious, the exponents $\{\frac{k_0-k_v}{2}\}_{v\mid\infty}$ in the determinant factors of $V_k(\mathbb{C})$ are integral. Therefore, on $\gamma\in B^\times$, $\bigotimes_v \mathrm{nrd}^{(\underline{k_0}-k_v)/2}$ evaluates to an element of F when k is paritious. Even when k is nonparitious, if $\gamma \in B^1$, then the determinant factors are all trivial. As such, the obstruction to the $\{T_\mathfrak{p}\}$ being defined over a finite extension is the field of definition of $\{\rho(\varpi_{j,h})\}_{j,h}$. and in particular that under $\bigotimes_v \mathrm{nrd}^{(\underline{k_0}-k_v)/2}$, $\{\varpi_{j,h}\}$ maps to some expression involving square roots.

To remedy this, define

$$V'_k(\mathbb{C}) := \bigotimes_{v\mid\infty} \mathrm{Sym}^{k_v-2}\,\mathbb{C}^2 = V_k(\mathbb{C})\otimes \mathrm{nrd}^{k-\underline{k_0}/2}.$$

By surjectivity of the reduced norm map to $\mathrm{Cl}^+_F = 1$ ([Voi21, 27.7.1, 28.5.5]), we may choose double coset representatives $\{\hat\alpha_h\}_{h=1}^H$ for $B^\times_+\backslash\widehat{B}^\times/\widehat{\mathcal{O}}^\times$ such that $\mathrm{nrd}(\hat\alpha) \in \hat{\mathbb{Z}}^\times_F$. With these choices in hand, Lemma 3.3 tells us that $\varpi_{j,h} \in \hat\alpha_{j^*(h)}\hat{\mathcal{O}}^\times\hat\pi_j\hat\alpha_l^{-1}\cap B^\times_+$. Therefore, $\mathrm{nrd}(\varpi_{j,h}) \in \mathrm{nrd}(\hat\alpha_{j^*(h)})\mathrm{nrd}(\hat\pi_j)\mathrm{nrd}(\hat\alpha_l^{-1})\cap F \ \in \mathfrak{p}$. Lemma 3.3 also tells us we can multiply $\varpi_{j,h}$ on the left by $\mathcal{O}^\times_{j^*(h)}$ without affecting $T_\mathfrak{p}$. Because $\epsilon\in\mathcal{O}^\times_{j^*(h)}$ for any $\epsilon\in\mathbb{Z}^\times_F$ and $\mathrm{Cl}^+_F = 1$, we can always realize $T_\mathfrak{p}$ using $\varpi_{j,h}$ such that $\mathrm{nrd}(\varpi_{j,h}) = \pi$ for any totally positive generator π of $\mathfrak{p}$. Evaluated on $\{\varpi_{j,h}\}$, the determinant factor in $V_k(\mathbb{C})$ is exactly $\pi^{\underline{k_0}-k/2}$. It follows that in the basis given by these $\{\hat\alpha_l\}$, T_π is given by the formulas in Lemma 3.3 but with the representation $V_k(\mathbb{C})$ replaced by $V'_k(\mathbb{C})$. The point is that by paying the cost of keeping track of the totally positive generator π of $\mathfrak{p}$, we have gotten rid of the determinant factors that were preventing the $\{T_\mathfrak{p}\}$ from being defined over a finite extension.

In particular, for any K over which the representation $V'_k(\mathbb{C})$ of $B^\times$ can be defined, the elemental Hecke operators $\{T_\pi\}$ can be defined over $K(\chi)$. We want to show that the $\{T_\pi\}$ can be defined over $F'(\chi)$.

We write $V'_k(K)$ for the representation $V'_k(\mathbb{C})$ thought of as a representation defined over K. For any $\alpha\in B^\times$, we can define the extension $K_\alpha := F(\sqrt{\alpha})$. Its Galois closure over $\mathbb{Q}$ is the multiquadratic extension $K^{\mathrm{gal}}_\alpha := F^{\mathrm{gal}}(\{\sqrt{\alpha_i}\})$, where $\{\alpha_i\}$ is the set of Galois conjugates of α. Taking α and β to elements whose norms are totally positive generators of distinct primes of F (possible by the Hasse-Schilling theorem, see e.g. [Voi21, Theorem 14.7.4]), we see K_α and K_β

split B and that $K_\alpha^{\mathrm{gal}} \cap K_\beta^{\mathrm{gal}} = F^{\mathrm{gal}}$. The representation $V_k'(\mathbb{C})$ can certainly be defined over both K_α^{gal} and over K_β^{gal}. Hence, the characteristic polynomials of $\{T_\pi\}$ – which are independent of the choice of basis – must be defined over $K_\alpha^{\mathrm{gal}}(\chi) \cap K_\beta^{\mathrm{gal}}(\chi) = F^{\mathrm{gal}}(\chi)$. However, we can do better. For any $\sigma \in \mathrm{Gal}(K_\alpha^{\mathrm{gal}}/\mathbb{Q})$ which fixes the weight k, $V_k'(K_\alpha^{\mathrm{gal}})$ is isomorphic to the representation obtained by applying σ entrywise. It follows that the characteristic polynomials of the $\{T_\pi\}$ are actually valued in $F'(\chi)$.

Because the elemental Hecke algebra is commutative and semisimple, the characteristic polynomials of $\{T_\pi\}$ being defined over $F'(\chi)$ implies that the matrices $\{T_\pi\}$ can themselves be descended to $F'(\chi)$ (see e.g. [Ser77, Chapter 12]).

□

Remark. We expect that it should be possible to give an argument using coherent cohomology, thinking of Hilbert modular forms as global sections of some automorphic line bundle. This approach has the advantage of treating partial weight one and higher weight forms uniformly. However, incorporating the determinant twists into the line bundle (and in particular, dealing with "square roots") seems subtle, and we elected to avoid these issues.

Applying Equation (16) to Equation (5), we find

$$T_{\pi^t} = T_\pi T_{\pi^{t-1}} - \pi^{k-\underline{1}}\chi^*(\mathfrak{p})T_{\pi^{t-2}} \qquad \text{and} \qquad T_{\gimel\mu} = T_\gimel T_\mu \text{ if } (\gimel, \mu) = 1. \tag{18}$$

Definition 5.4. The elemental Hecke algebra, $\mathbb{T}^{\mathrm{elem}} := \mathbb{T}^{\mathrm{elem}}_{F'(\chi)}$, is the $F'(\chi)$ algebra generated by the $\{T_\pi\}$.

When k is paritious, $\mathbb{T}^{\mathrm{elem}}_{F'(\chi)} \cong \mathbb{T}_{F'(\chi)}$, and even when k is nonparitious, $\mathbb{T}^{\mathrm{elem}}_{\overline{\mathbb{Q}}} \cong \mathbb{T}_{\overline{\mathbb{Q}}}$. $\mathbb{T}$-submodules (resp. $\mathbb{T}$-eigenforms) are the same as $\mathbb{T}^{\mathrm{elem}}$-submodules (resp. $\mathbb{T}^{\mathrm{elem}}$-eigenforms).

For a given totally positive generator δ of $\mathfrak{d}_F$, we say that a $\mathbb{T}^{\mathrm{elem}}$-eigenform f is δ-normalized if $a_{\delta^{-1}}(f) = 1$.

Lemma 5.5. *For $\mathbb{T}^{\mathrm{elem}}$ the elementary Hecke algebra on $S_k(\mathfrak{N}, \chi)$ for $k \in \mathbb{Z}^n_{\geq 2}$, there is a bijection*

$$\begin{aligned} \Phi\colon \ \big\{\delta\text{-normalized } \mathbb{T}^{\mathrm{elem}} \text{ eigenforms}\big\} &\longrightarrow \mathrm{Hom}_{F'(\chi)\text{-alg}}(\mathbb{T}^{\mathrm{elem}}, \mathbb{C}) \\ f &\longmapsto (T \longmapsto a_{\delta^{-1}}(Tf)) \end{aligned}.$$

The map $\Phi(f)$ sends T_μ to the coefficient $a_{\delta^{-1}\mu}(f)$, and the field extension of $F'(\chi)$ generated by the image of $\Phi(f)$ is exactly the extension generated by the coefficients of f.

Proof. For a δ-normalized eigenform f, $a_{\delta^{-1}}(T_\mu(f)) = a_{\delta^{-1}\mu}(f)$. The proof of Lemma 5.5 is then the same as that of the analogous fact for classical modular forms. □

This proves finiteness of the extension $\mathbb{Q}(\{a_\nu(f)\})$ for f a δ-normalized eigenform. Because $\mathbb{T}^{\mathrm{elem}}$ is a finite commutative $F'(\chi)$-algebra, there exist field extensions $K_f/F'(\chi)$ such that $\mathbb{T}^{\mathrm{elem}} \cong \prod_f K_f$. As in Section 4.1, we can write $S_k(\mathfrak{N}, \chi)^{\mathrm{new}} \cong \bigoplus_f V_f$ where each V_f is an irreducible $\mathbb{T}^{\mathrm{elem}}$-submodule on which $\mathbb{T}^{\mathrm{elem}}$ acts by $K_f/F'(\chi)$. Indeed, because $\mathrm{Hom}_{F'(\chi)\text{-alg}}(\mathbb{T}^{\mathrm{elem}}, \mathbb{C})$ is preserved by post-composition with automorphisms of $\mathbb{C}$ fixing $F'(\chi)$, it follows that the sum is indexed by Galois orbits of newforms.

We can then obtain a basis of V_f defined over $F'(\chi)$ exactly as in Section 4.1, by taking the orbit of a trace form $g \in V_f$ under some generator of $\mathbb{T}^{\mathrm{elem}}|_{V_f}$. We can replace Lemma 4.2 with

$$a_{\delta^{-1}\nu}(T^j g) = \operatorname{tr}(T^j T_\nu). \tag{19}$$

We want to show that all of $S_k(\mathfrak{N}, \chi)$ (not just the new subspace) has a basis over $F'(\chi)$. Applying Equation (3) to Equation (6), and writing ξ for a totally positive generator of $\mathfrak{D}$, we find

$$a_\nu(\iota_{\mathfrak{D}}(f)) = \xi^{(k-\underline{k_0})/2} a_{\nu\xi^{-1}}(f).$$

Because our aim is to produce a basis for $S_k(\mathfrak{N}, \chi)$, we do not care about multiplicative scalar factors. As such we can define $\iota'_\xi(f)$ by $a_\nu(\iota'_\xi(f)) := a_{\nu\xi^{-1}}(f)$. The image of $\iota'_\xi \otimes \mathbb{C}$ is the same as the image of $\iota_\xi \otimes \mathbb{C}$, so it does not affect the complex space we produce in the end.

Theorem 5.6. *The space $S_k(\mathfrak{N}, \chi)$ has a basis over $F'(\chi)$.*

Proof. First, assume that $k \in \mathbb{Z}^n_{\geq 2}$. We have

$$S_k(\mathfrak{N}, \chi) \cong \bigoplus_{\mathfrak{M}|\mathfrak{N}} \bigoplus_{\mathfrak{D}|\mathfrak{N}\mathfrak{M}^{-1}} \bigoplus_{\substack{\text{newform orbits } f \\ \text{of level } \mathfrak{M}}} \iota'_\xi(V_f), \tag{20}$$

where in the summand, ξ is a totally positive generator of $\mathfrak{D}$. As noted above, each V_f has a basis of forms over $F'(\chi)$, and if $g \in V_f$ has coefficients in $F'(\chi)$, so does $\iota'_\xi(g)$.

Suppose instead that k is of partial weight one. By [BB66, Theorem 10.11], for large enough $l \in \mathbb{Z}_{\geq 2}$, the space of parallel weight l forms $M_{\underline{l}}(\mathfrak{N}, 1)$ contains a set of forms with no common zeroes on $X_0(\mathfrak{N})$. Since $M_{\underline{l}}(\mathfrak{N}, 1)$ has a basis $\{g_1, \ldots, g_d\}$ over $\mathbb{Q}$, the elements of this basis must then have no common zeroes (as otherwise everything in their span would share a zero). Now consider the map

$$\begin{aligned} \Theta\colon\ S_k(\mathfrak{N}, \chi) &\longrightarrow S_{k+\underline{l}}(\mathfrak{N}, \chi\psi)^d \\ f &\longmapsto (fg_1, \ldots, fg_d). \end{aligned}$$

We claim that the image of Θ is exactly the subspace of $h = (h_1, \ldots, h_d) \in S_{k+\underline{l}}(\mathfrak{N}, \chi\psi)^d$ where $g_i h_j = g_j h_i$ for all $i, j \in [d]$. For any $f \in S_k(\mathfrak{N}, \chi)$, $\Theta(f)$ is in this subspace. Conversely, given h in the subspace, $\frac{h_i}{g_i}$ is independent of i. This quotient is holomorphic because for any $z \in X_0(\mathfrak{N})$, some $\operatorname{ord}_z(g_i) \neq 0$ and so some $\operatorname{ord}_z(h_i/g_i) = \operatorname{ord}_z(h_i) \geq 0$.

We already know that $S_{k+\underline{l}}(\mathfrak{N}, \chi\psi)$ has a basis over $F'(\chi)$, and we have just shown that $S_k(\mathfrak{N}, \chi)$ can be identified with a subspace of $S_{k+\underline{l}}(\mathfrak{N}, \chi\psi)^d$ cut out by equations in $F'(\chi)$. Therefore, $S_k(\mathfrak{N}, \chi)$ also has a basis of q-expansions over $F'(\chi)$. $\square$

Remark. Theorem 5.6 is tight. The field of coefficients K_f of $f \in S_k(\mathfrak{N}, \chi)$ always needs to include the field of definition of χ. Because $a_{\epsilon\nu}(f) = \epsilon^{k/2} a_\nu(f)$, K_f also needs to include $\epsilon^{k/2}$ for all $\epsilon \in \mathbb{Z}^\times_{F,>0}$. As discussed earlier, totally positive units in

F are squares since $\mathrm{Cl}_F^+ \cong \mathrm{Cl}_F$. As such, $\epsilon^{k/2}$ is an element of F^{gal}. Picking a unit ϵ such that $\mathbb{Q}(\epsilon^{1/2}) = F$, we deduce that K_f also contains F', and hence contains $F'(\chi)$.

Theorem 5.7. *The elemental Hecke operators* $\{T_\pi\}$ *on* $S_k(\mathfrak{N}, \chi)$ *are defined over* $F'(\chi)$ *for any* $k \in \mathbb{Z}_{\geq 1}^n$.

Proof. If $k \in \mathbb{Z}_{\geq 2}^n$, the result follows from Lemma 5.3. Otherwise, we still know that $S_k(\mathfrak{N}, \chi)$ has a basis of forms with coefficients in $F'(\chi)$ by Theorem 5.6. The elemental Hecke operators on $S_k(\mathfrak{N}, \chi)$ are given by Equation (17), and in particular $T_\pi f$ has coefficients in $F'(\chi)$ if f does $F'(\chi)$. It follows that in this basis, T_π is given by a matrix with entries in $F'(\chi)$. □

From this and Lemma 5.5, we can use the arguments that we used in the $\mathbb{Z}_{\geq 2}^n$ case to deduce the following theorems.

Proposition 5.8. *For any* $k \in \mathbb{Z}_{\geq 1}^n$*, let* f *be a* δ*-normalized* $\mathbb{T}^{\mathrm{elem}}$*-eigenform in* $S_k(\mathfrak{N}, \chi)$*. The extension* $\mathbb{Q}(\{a_\nu(f)\})$ *is finite.*

Proposition 5.9. *If* $f \in S_k(\mathfrak{N}, \chi)$ *with* $k \in \mathbb{Z}_{\geq 1}^n$ *and* τ *is an automorphism of* $\mathbb{C}$ *fixing* $F'(\chi)$*, then*

$$^{\tau}f(z) := \sum_{\nu \in \mathfrak{d}_{F,>0}^{-1}} \tau(a_\nu(f)) \exp\left(2\pi i \sum_j \iota(\nu_j) z_j\right)$$

is an element in $S_k(\mathfrak{N}, \chi)$*. Furthermore, if* f *is a Hecke eigenform, then so is* $^{\tau}f$*.*

Observe that Theorem 5.7, Proposition 5.8, and Proposition 5.9 are strict strengthenings of Theorem 2.2, Proposition 2.4, and Proposition 2.5.

5.3. **Computing spaces of forms in nonparitious weight.** With the theory of Section 5.2 in hand, we can compute spaces of nonparitious forms with $k \in \mathbb{Z}_{\geq 2}^n$ by replacing $T_\mathfrak{p}$ with T_π (for some totally positive generator π of $\mathfrak{p}$) and $a_\mathfrak{n}$ with a_ν everywhere. By doing this, we are able to work with spaces and matrices over $F'(\chi)$ instead of dealing with field extensions depending on $\mathfrak{p}$. We walk through the steps of Section 4 and highlight the modifications that need to be made.

1. **Compute "full" Hecke matrices:** To compute T_π, we repeat the procedure in Section 3, choosing $\{\varpi_{j,h}\}$ in Lemma 3.3 whose norms are all equal to π and replacing $V_k(\mathbb{C})$ with $V_k'(\mathbb{C})$ (i.e. forgetting about the determinant factors). The determinant factor in $V_k(\mathbb{C})$ for these $\{\varpi_{j,h}\}$ is simply a twist by $\pi^{k_0 - k/2}$ (independent of j and h). As $T_\mathfrak{p} = \pi^{k_0 - k/2} T_\pi$, removing the determinant factors lets us produce a matrix for T_π.
2. **Restrict the Hecke matrices to Galois orbits of newforms:** Because each T_π is a rescaling of $T_\mathfrak{p}$, we can replace $T_\mathfrak{p}$ with T_π in Proposition 4.1 without changing the subspace we produce. Let T be a generator $T \in \mathbb{T}^{\mathrm{elem}}|_{S_k(\mathfrak{N},\chi)^{\mathrm{new}}}$, and μ_T its characteristic polynomial. As discussed in Section 5.2, the elemental Hecke algebra still decomposes as a product of fields K_f, so Equation (14) still holds. Therefore, we can decompose the new subspace as a direct sum of the kernels of the evaluations at T of the factors of μ.
3. **Produce matrices for T_μ from the matrices for T_π:** We can compute these efficiently using Equation (18) and dynamic programming.

Table 1. Some Fourier coefficients of an eigenform in $S_k(\mathfrak{N}, \chi)$ for k, $\mathfrak{N}$, and χ given above. Here, π is a totally positive generator of a prime ideal $\mathfrak{p}$, δ is the chosen generator for the different of $\mathbb{Z}_F$, and α is a generator for the coefficient field $K = \mathbb{Q}[x]/(x^4 + 24x^2 + 46)$ such that $\alpha^2 = 7\sqrt{2} - 12$.

Nm($\mathfrak{p}$)	π	$\mathbf{a}_{\pi\delta^{-1}}(\mathbf{f})$
2	$-\sqrt{2}+2$	$-\alpha$
7	$-3\sqrt{2}+5$	$\frac{1}{7}(-2\alpha^3+6\alpha^2-31\alpha-5)$
7	$-\sqrt{2}+3$	$\frac{1}{7}(\alpha^3+26\alpha)$
9	3	$\frac{1}{7}(3\alpha^3+106\alpha)$
17	$-2\sqrt{2}+5$	$\frac{1}{7}(-8\alpha^2-222)$
17	$-4\sqrt{2}+7$	$\frac{1}{7}(3\alpha^3+120\alpha)$
23	$-\sqrt{2}+5$	$\frac{1}{7}(26\alpha^2+564)$
23	$-7\sqrt{2}+11$	$\frac{1}{7}(36\alpha^2+26)$
25	5	$\frac{1}{7}(2\alpha^3-74\alpha)$
31	$-5\sqrt{2}+9$	$\frac{1}{7}(13\alpha^3+30\alpha)$
31	$-3\sqrt{2}+7$	$\frac{1}{7}(-30\alpha^2-122)$

(4) **Compute q-expansions of an $F'(\chi)$-basis of each newform orbit:** This was discussed in the proof of Theorem 5.6. The key point is that we use Equation (19) in lieu of Lemma 4.2.

(5) **Assemble the bases of newform orbits to produce a basis for $S_k(\mathfrak{N}, \chi)$:** This was discussed in the proof of Theorem 5.6, and can be done using Equation (20).

In the case of partial weight one, the Hecke stability method of Section 4.2 can be applied almost verbatim. We are able to compute bases of spaces in weights $\mathbb{Z}^n_{\geq 2}$ as just described, and compute Hecke operators on the space of modular quotients using Equation (17) instead of Equation (4).

6. Examples

We leave the reader with the following two examples of our code. The codebase can be found here.

6.1. **A weight $(4, 3)$ space over $\mathbb{Q}(\sqrt{2})$.** In this subsection, we reproduce a computation of [DLP19].

Let $k := (4, 3)$, $\mathfrak{N} = (\sqrt{2} + 3)|7$, and χ the nontrivial ray class character unramified away from $\mathfrak{N}$ and the infinite places. Then, $S_k(\mathfrak{N}, \chi)$ is two-dimensional. Choosing the generator $\delta := -2\sqrt{2} + 4$, the coefficients of one of the two conjugate δ-normalized eigenforms in this space are given in Table 1. The coefficients of f lie in $K = \mathbb{Q}[x]/(x^4 + 24x^2 + 46)$.

The coefficients of $a_{\pi\delta^{-1}}(f)$ and for $\mathrm{Nm}(\pi) \leq 200$ can be computed in under 7 seconds on a single core of an 11th Gen Intel(R) Core(TM) i7-11800H @ 2.30 GHz (my laptop) – the code is given below. Of course, there is a lot going on under the hood! Nonetheless, we feel that one of the merits of this work is that it makes such computations accessible "at the push of a button".

```
// specify the field, level, weight, and nebentypus
```

```
F := QuadraticField(2);
ZF := Integers(F);
k := [4, 3];
N := Factorization(7*ZF)[1][1];
H := HeckeCharacterGroup(N, [1,2]);
chi := H.1;

// controls how many coefficients we compute
BOUND := 200;

// set up the relevant space of Hilbert modular forms
M := GradedRingOfHMFs(F, BOUND);
Mk := HMFSpace(M, N, k, chi);

// compute an basis of q-expansions spanning the space
// the basis will be over F'(chi) = F
Sk := CuspFormBasis(Mk);

// the dimension of the cusp space is 2
assert #Sk eq 2;

// diagonalize the Hecke action to produce an eigenbasis
eigs := Eigenbasis(Mk, Sk : P:=10);
```

6.2. **Weight $(1,2)$ forms over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$.** This project was motivated by a proposal ([Cal21]) for computing explicit examples of 4-folds of Mumford's type by computing Hilbert modular forms that are expected to be associated to them. These Hilbert modular forms are expected to be non-CM forms of weight $(1,1,2)$ and have coefficients satisfying certain rationality conditions. One also expects a similar geometric origin for non-CM forms of weight $(1,2)$ with coefficients satisfying similar rationality conditions. While do we not yet have any examples of non-CM forms, we report on some of our findings thusfar.

Given k, $\mathfrak{N}$, and χ, we can compute the space of CM forms $D_k(\mathfrak{N},\chi) \subseteq S_k(\mathfrak{N},\chi)$ by searching through the CM extensions of F with conductor dividing $\mathfrak{N}$ and looking for Hecke characters with an infinity type dependent on k and behavior at finite places determined by χ. Given such a Hecke character, one can produce explicit formulas for the Fourier coefficients of the corresponding automorphically induced Hilbert modular form. As such, we can verify whether a given form is CM or not by checking to see if it lands in the CM space.

Theorem 6.1. *The weight $(1,2)$ forms over $F = \mathbb{Q}(\sqrt{2})$ and $F = \mathbb{Q}(\sqrt{5})$ of Galois stable level $\mathfrak{N}$ with $\mathrm{Nm}(\mathfrak{N}) \leq 1500$ and quadratic nebentypus character are all CM forms.*

It may seem odd to focus on Galois stable level. While there are indications that forms of Galois stable level might relate more easily to geometry, the reason in our setting is practical. For such levels $\mathfrak{N}$, one can check by a class field theory computation that unless $\mathfrak{N}|(2)$, there will always exist Eisenstein series $g \in M_{\underline{1}}(\mathfrak{N},\psi)$ which are nonvanishing at the cusp at infinity. While the Hecke stability method will work with Eisenstein series of any weight, the bottleneck in the computation of $(1,2)$ forms is computing a basis of q-expansions for a space of weight $(1+k_g, 2+k_g)$, where k_g is the weight of the Eisenstein series.

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