COMPUTING HILBERT MODULAR FORMS WITH NONTRIVIAL NEBENTYPUS

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Abstract. We describe algorithms for computing matrices for the Hecke action on spaces of Hilbert modular forms with nontrivial nebentypus.

1. Introduction

Let F be a totally real field of degree n, with ring of integers $\mathbb{Z}_F$. For our purposes, a space of Hilbert modular forms over F is specified by a level $\mathfrak{N} \subset \mathbb{Z}_F$, a weight $k \in \mathbb{Z}^n_{\geq 1}$, and a nebentypus character χ of modulus $\mathfrak{N}$ (see **??**) – we denote this space by $M_k(\mathfrak{N}, \chi)$. We write $S_k(\mathfrak{N}, \chi)$ for the space of cusp forms in $M_k(\mathfrak{N}, \chi)$. For any prime $\mathfrak{p} \subset \mathbb{Z}_F$, there is an action of the Hecke operator $T_\mathfrak{p}$ on $S_k(\mathfrak{N}, \chi)$. By the Jacquet-Langlands correspondence [**?**], we can compute these matrices by studying the Hecke action on certain spaces of quaternionic modular forms for a quaternion algebra B/F. Building on work of Eichler [**?**,**?**], Shimura [**?**], Pizer [**?**], and others, algorithms for producing these matrices when F has narrow class number 1, $k = (2, \dots, 2)$ (parallel weight two), and $\chi = 1$ (trivial nebentypus) were invented and implemented by Greenberg-Voight [**?**] for indefinite B and by Dembélé [**?**] for definite B. These methods were extended to the fields of arbitrary narrow class number and general paritious weights by Voight [**?**] and Dembélé-Donnelly [**?**] respectively. In this note, we describe algorithms for computing matrices for the Hecke action on spaces of Hilbert modular forms with paritious weight and nontrivial nebentypus. These algorithms and their proofs will likely be unsurprising to the experts, and in many cases they are natural generalizations of the corresponding results for classical modular forms. Nevertheless, we felt that it would be beneficial to provide self-contained proofs of these generalizations.

Throughout, we recall the notions and notations for Hecke characters and for Hilbert modular forms from [**?**, Section 2].

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2. Quaternionic modular forms as automorphic forms

Let B/F be a quaternion algebra split at the first r infinite places and ramified at the last $s := n - r$ (we can always reorder the places of F so that this holds). Fix a maximal compact subgroup K_∞ of $B^\times_\infty := B \otimes_F \otimes_{v|\infty} F_v \cong (\mathrm{GL}_2(\mathbb{R}))^r \times (\mathbb{H}^\times)^s$, and at each finite place v fix a hyperspecial compact subgroup K_v of $B^\times_v$. With these choices in hand, we may define the group $B^\times(\mathbb{A}_F) := \prod'_v (B^\times_v; K^\times_v)$. Let $\widehat{K} := \prod_{v \nmid \infty} K_v$. Then, $K^\times_\infty \widehat{K}^\times$ is a maximal compact subgroup of $K^\times$. We can define a maximal order $\mathcal{O}_0(1) := B \cap \widehat{K}$, where B embeds into $B^\times(\mathbb{A}_F)$ diagonally. An automorphic form for $B^\times(\mathbb{A}_F)$ with central character $\omega \colon \mathbb{A}^\times_F \to S^1$ is a function

$$\varphi \colon B^\times \backslash B^\times(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

that is K-finite (the span of right translates of ϕ by $k \in K$ is finite-dimensional), has central character ω, and satisfies a moderate growth condition. Fix an order $\mathcal{O} \subset \mathcal{O}_0(1)$ and semigroup homomorphism $\Xi \colon \hat{\mathcal{O}} \cap \widehat{B}^\times \to \mathbb{C}$ such that $\Xi|_{\mathbb{A}^\times_F \cap \hat{\mathcal{O}}} = \omega|_{\mathbb{A}^\times_F \cap \hat{\mathcal{O}}}$. We call the conductor of Ξ the product of primes at which the local homomorphism $\Xi_\mathfrak{p}$ is nontrivial on $\widehat{\mathcal{O}}_\mathfrak{p} \cap \widehat{B}^\times_\mathfrak{p}$.

Example 2.1. Consider the automorphic forms corresponding to classical modular forms of level N and nebentypus a finite-order Hecke character χ. In this case, $B = M_{2,\mathbb{Q}}$ and $\widehat{\mathcal{O}} = K_0(N) \subset M_2(\widehat{\mathbb{Z}})$ consists of the matrices which are upper trianglular modulo N. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widehat{\mathcal{O}}$. Abusing notation, we obtain a semigroup homomorphism

$$\begin{aligned} \chi_0 \colon\ M_2(\hat{\mathbb{Z}}) &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto \chi_0(d) \end{aligned}.$$

If $\mathcal{O}$ is an Eichler order $\mathcal{O}_0(\mathfrak{N})$ and B is split at every prime dividing $\mathfrak{N}$, any character $\chi_0 \colon (\mathbb{Z}_F/\mathfrak{N})^\times \to \mathbb{C}^\times$ corresponds to a semigroup homomorphism on $\widehat{\mathcal{O}_0(\mathfrak{N})}$ exactly as in Example 2.1, which we also denote by χ_0.

Remark. It is necessary to specify Ξ and ω separately. For example, suppose that $\mathcal{O} = \mathcal{O}_0(\mathfrak{N})$. Then, $\mathbb{A}_F^\times \cap \widehat{\mathcal{O}} = \prod_{v\nmid\infty} \mathbb{Z}_{F,v}^\times =: \widehat{\mathbb{Z}}_F^\times$, so Ξ determines ω_0 in the language of [?, Section 2]. The unitary Hecke characters ω extending $\omega_0 = \Xi|_{\widehat{\mathbb{Z}}_F}^\times$ form a $(\mathrm{Cl}_F^+)^\vee$-torsor, since we can multiply any such extension ω by a character of Cl_F^+ to get another extension ω' with $\omega_0 = \omega_0'$. For general $\mathcal{O}$, $\mathbb{A}_F^\times \cap \widehat{\mathcal{O}}$ can be even smaller, and the gap between Ξ and ω may be even more pronounced.

We say that an automorphic form φ has level $\mathcal{O}$ and nebentypus Ξ if for any $\hat{u} \in \widehat{\mathcal{O}}^\times$, $\varphi(x\hat{u}) = \Xi(\hat{u})\varphi(x)$. Let

$$B_{\infty,\mathbb{R}}^\times := \prod_{\substack{v\mid\infty \\ v \text{ split}}} B_v^\times \cong \mathrm{GL}_2(\mathbb{R})^r \qquad \text{and} \qquad B_{\infty,\mathbb{C}}^\times := \prod_{\substack{v\mid\infty \\ v \text{ ramified}}} B_v^\times \cong (\mathbb{H}^\times)^s.$$

Similarly, let $K_{\infty,\mathbb{R}} := K_\infty \cap B_{\infty,\mathbb{R}} \cong O_2(\mathbb{R})^r$ and $K_{\infty,\mathbb{C}} := K_\infty \cap B_{\infty,\mathbb{C}} \cong \mathrm{SU}_2(\mathbb{R})^s$. A quaternionic modular form is an automorphic form for $B^\times$ satisfying an additional condition at the infinite real places.

Definition 2.2. A quaternionic modular form for $B^\times$ of level $\mathcal{O}$ and nebentypus Ξ is a function

$$\varphi \colon B^\times \backslash B^\times(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

such that

(1) For any $\hat{u} \in \widehat{\mathcal{O}}^\times$, $\varphi(x\hat{u}) = \Xi(\hat{u})\varphi(x)$;
(2) ϕ is K-finite;
(3) The automorphic representation associated to ϕ is holomorphic discrete series. In particular, for any $\kappa = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}^\epsilon \in K_{\infty,\mathbb{R}}$,

$$\varphi(x\kappa) = \frac{j(\kappa, \sqrt{-1})^k}{|\det \kappa|^{k/2}} \varphi(x) = (-1)^{\epsilon k} e^{\sqrt{-1}\theta} \phi(x). \tag{1}$$

The space of such forms is denoted $M_k^B(\mathcal{O}, \Xi)_{\mathrm{aut}}$. The subspace of $M_k^B(\mathcal{O}, \Xi)_{\mathrm{aut}}$ with central character ω is denoted $M_k^B(\mathcal{O}, \Xi, \omega)_{\mathrm{aut}}$.

As noted earlier, there is a decomposition

$$M_k^B(\mathcal{O}, \Xi)_{\mathrm{aut}} \cong \bigoplus_{\omega} M_k^B(\mathcal{O}, \Xi; \omega) \tag{2}$$

where ω ranges over Hecke characters of F extending $\Xi|_{\mathbb{A}_F^\times \cap \widehat{\mathcal{O}}}$.

We now define the Hecke action on $M_k^B(\mathcal{O}, \Xi, \omega)_{\mathrm{aut}}$. Given a prime ideal $\mathfrak{p} \subset \mathbb{Z}_F$, let $\hat{\pi} \in \widehat{B}^\times$ be an element which at places $v \neq \mathfrak{p}$ is 1 and at $v = \mathfrak{p}$ is an element whose reduced norm is a uniformizer for $F_\mathfrak{p}$. Equivalently, we may choose $\hat{\pi}$ such that $\mathrm{nrd}(\hat{\pi})\hat{\mathbb{Z}}_F \cap F = \mathfrak{p}$. There exists a $P \in \mathbb{Z}$ and elements $\{\hat{\pi}_j\}_{j=1}^P \subset \widehat{B}^\times$ such that

$$\widehat{\mathcal{O}}^\times \backslash \widehat{\mathcal{O}}^\times \hat{\pi} \widehat{\mathcal{O}}^\times = \bigsqcup_{j=1}^P \widehat{\mathcal{O}}^\times \hat{\pi}_j. \tag{3}$$

When $\mathcal{O}$ is $\mathcal{O}_0(\mathfrak{N})$, P is $\mathrm{Nm}(\mathfrak{p}) + 1$ if $\mathfrak{p} \nmid \mathfrak{N}$ and $\mathrm{Nm}(\mathfrak{p})$ otherwise.

We now define

$$(T_{\mathfrak{p}}\varphi)(x) := \sum_{j=1}^{P} \varphi(x\hat{\pi}_j^{-1})\Xi(\hat{\pi}_j). \tag{4}$$

This definition comes from the theory of automorphic representations (see e.g. [?, Chapter 3]). By the first property in Definition 2.2, Equation (4) depends only on the cosets $\{\widehat{\mathcal{O}}^\times\hat{\pi}_j\}$ and not on the specific representatives $\{\hat{\pi}_j\}$. Definition 2.2 and Equation (4) together define the space of quaternionic modular forms as a Hecke module.

There is a distinguished subspace of $M_k^B(\mathcal{O},\Xi)$ which are spanned by the functions $\phi_\eta := \eta \circ \mathrm{nrd}$ for η a Hecke character of F. For such a function, $\eta(\mathrm{nrd}(\hat{u})) = \phi(\hat{u}) = \Xi(\hat{u})$, so we must have $\eta|_{\mathrm{nrd}(\widehat{\mathcal{O}}^\times)} = \Xi$. The function ϕ_η has central character η^2. We write $\mathcal{E}_k^B(\mathcal{O},\Xi,\omega)$ for the subspace of $M_k^B(\mathcal{O},\Xi,\omega)$ spanned by the $\{\phi_\eta\}$, and $S_k^B(\mathcal{O},\Xi,\omega)$ for the complement of $\mathcal{E}_k^B(\mathcal{O},\Xi,\omega)$. Even though the Shimura variety associated to B has no cusps when $B \neq M_2(F)$, we nonetheless call $S_k^B(\mathcal{O},\Xi,\omega)$ the space of quaternionic cusp forms[1].

2.1. **The Jacquet-Langlands correspondence and Hilbert modular forms.** Our primary motivation for computing spaces of quaternionic modular forms is that they are isomorphic to certain spaces of Hilbert modular forms. Let B be a quaternion algebra unramified outside the infinite places.

Given an ideal $\mathfrak{D} \subset \mathbb{Z}_F$ and a space of Hilbert cusp forms $S_k(\mathfrak{N},\Xi)$, the $\mathfrak{D}$-new subspace of $S_k(\mathfrak{N},\Xi)$ is the complement of the images of the degeneracy maps at $\mathfrak{p}$ for all primes $\mathfrak{p}|\mathfrak{D}$.

Theorem 2.3 (Eichler-Shimizu-Jacquet-Langlands (see e.g. [?, Theorem 3.9]). *Let $\mathfrak{N} \subset \mathbb{Z}_F$ be a level, $k \in \mathbb{Z}_{\geq 2}^n$ a weight, and χ a finite-order Hecke character. Let B/F be a quaternion algebra with discriminant $\mathfrak{D}$ such that* $\mathrm{cond}(\chi)|\mathfrak{N}\mathfrak{D}^{-1}$. *Then, there is a Hecke module isomorphism*

$$S_k^B(\mathcal{O}_0(\mathfrak{N}\mathfrak{D}^{-1}),\chi_0) \cong S_k(\mathfrak{N},\chi)^{\mathfrak{D}\text{-new}}.$$

By Theorem 2.3, we can at least access Hecke matrices on the $\mathfrak{D}$-new subspace of $S_k(\mathfrak{N},\chi)$ by computing Hecke matrices on $S_k^B(\mathfrak{N},\chi)$. In particular, if the discriminant of B is (1), then $S_k^B(\mathcal{O}_0(\mathfrak{N}),\chi) \cong S_k(\mathfrak{N},\chi)$ on the nose by Theorem 2.3. Given any subset of places of F of even cardinality, there exists a quaternion algebra B/F ramified at exactly those places. When n is even, we can choose B in to be a definite quaternion algebra ramified at every infinite place ($r = 0$) and unramified away from the infinite places. When n is odd, we can choose B to be an indefinite quaternion algebra ramified at all but one of the infinite places ($r = 1$) and unramified away from the infinite places. It follows that for any totally real field, we can choose a B for which the Hecke matrices on $S_k^B(\mathcal{O}_0(\mathfrak{N}),\chi)$ are exactly the Hecke matrices on $S_k(\mathfrak{N},\chi)$.

Because $S_k(\mathfrak{N},\chi)$ is spanned by Hecke eigenforms, a basis of q-expansions for $S_k(\mathfrak{N},\chi)$ can be recovered from Hecke matrices on $S_k^B(\mathfrak{N},\chi)$ for sufficiently many primes $\mathfrak{p} \subset \mathbb{Z}_F$. This was made algorithmic by [?]. As such, the problem of computing a basis of $S_k(\mathfrak{N},\chi)$ reduces to computing Hecke matrices on $S_k^B(\mathcal{O}_0(\mathfrak{N}),\chi)$.

Unfortunately, the description of quaternionic modular forms given by Definition 2.2 and Equation (4) is not amenable to explicitly computing Hecke matrices. In what follows, we will deduce from Definition 2.2 and Equation (4) several equivalent descriptions of the space of quaternionic modular forms as a Hecke module, and ultimately give algorithms for computing Hecke matrices on $S_k^B(\mathcal{O},\Xi)$.

Theorem 2.4. *Given a totally real field F of degree n, a prime ideal $\mathfrak{p} \subset \mathbb{Z}_F$, a quaternion algebra B/F ramified at all or all but one of the infinite places of F, a paritious weight $k \in \mathbb{Z}_{\geq 2}^n$, an order $\mathcal{O} = \mathcal{O}_0(\mathfrak{N}) \subset B$, and a semigroup homomorphism $\chi\colon \widehat{\mathcal{O}} \cap \widehat{B}^\times \to \mathbb{C}$ that is trivial on $\mathcal{O}_v$ for any ramified finite place v, there is an algorithm that produces a matrix for the action of the Hecke operator $T_{\mathfrak{p}}$ on $S_k^B(\mathcal{O},\chi)$, in a basis independent of $\mathfrak{p}$.*

The methods in this paper will actually work for any $\mathcal{O}$, but some of the algorithms in [?] which we call as subroutines assume $\mathcal{O}$ to be of this form. We expect this requirement can be relaxed without much difficulty, but do not treat this here. Combining Theorem 2.4 with the algorithms of [?] and the Hecke stability method (described in [?] for modular forms and extended in [?, ?] to Hilbert modular forms) for computing partial weight 1 forms, we deduce the following.

[1]The forms in $\mathcal{E}_k^B(\mathcal{O},\Xi,\omega)$ are sometimes called quaternionic Eisenstein series. However, I find this terminology confusing as the representations associated to these forms are not in the continuous spectrum.

Corollary 2.5. *Let F be a totally real field. Given an integral ideal $\mathfrak{N}$ of F, a paritious weight $k \in \mathbb{Z}_{\geq 2}^{[F:\mathbb{Q}]}$, and a finite order Hecke character χ of F, there is an algorithm that computes a basis of q-expansions (to any given precision) spanning $S_k(\mathfrak{N}, \chi)$.*

3. The "usual" definition of quaternionic modular forms

The definition of $M_k^B(\mathcal{O}, \Xi)$ that one sees more often (e.g. [?, Definition 7.5]) is the following.

Definition 3.1. The space of quaternionic modular forms of level $\mathcal{O}$ and nebentypus Ξ consists of the functions

$$\phi \colon \mathcal{H}^r \times \widehat{B}^\times \longrightarrow W_k$$

such that

(1) For any $\gamma \in B_+^\times$,

$$\phi(z, \hat{x}) = \prod_{i=1}^{r} \frac{\det \gamma_i^{k_i/2}}{j(\gamma_i, z_i)^{k_i}} \phi(\gamma z, \gamma \widehat{\mathcal{O}}^\times)^\gamma.$$

(2) For any $\hat{u} \in \widehat{\mathcal{O}}^\times$,

$$\phi(z, \hat{z}\hat{u}) = \Xi(\hat{u})\phi(z, \hat{z}).$$

(3) ϕ is holomorphic in the first variable.

We denote this space by $M_k^B(\mathcal{O}, \Xi)$, and the subspace of $M_k^B(\mathcal{O}, \Xi)$ with central character ω by $M_k^B(\mathcal{O}, \Xi, \omega)$.

We can define the Hecke operator $T_\mathfrak{p}$ on ϕ as in Definition 3.1 by the formula

$$T_\mathfrak{p}\phi(z, \hat{x}) = \sum_{j=1}^{P} \phi(z, \hat{x}\hat{\pi}_j^{-1})\Xi(\hat{\pi}_j). \tag{5}$$

As with Equation (4), the right-hand side of Equation (5) is independent of the choices of $\{\hat{\pi}_j\}$. Here is the main result of this section.

Proposition 3.2. *As Hecke modules, $M_k^B(\mathcal{O}, \Xi, \omega) \cong M_k^B(\mathcal{O}, \Xi, \omega)_{\mathrm{aut}}$.*

We will prove Proposition 3.2 by defining a third space $M_k^B(\mathcal{O}, \Xi, \omega)'_{\mathrm{aut}}$ and Hecke module isomorphisms $M_k^B(\mathcal{O}, \Xi, \omega)_{\mathrm{aut}} \xrightarrow{\sim} M_k^B(\mathcal{O}, \Xi, \omega)'_{\mathrm{aut}} \xrightarrow{\sim} M_k^B(\mathcal{O}, \Xi, \omega)$.

Let $\varphi \in M_k^B(\mathcal{O}, \Xi)_{\mathrm{aut}}$ be a quaternionic modular form as in Definition 2.2. Consider the space W spanned by the right translates of φ by $(\mathbb{H}^\times)^s$. As a Lie group, $\mathbb{H}^\times \cong \mathrm{SU}_2(\mathbb{R}) \times \mathbb{R}_{>0}^\times$ – the former is compact and the latter is in the center. Because φ is K-finite and has a central character, W is a twist of a finite-dimensional irreducible representation of $\mathrm{SU}_2(\mathbb{R})^s$ by a power of the determinant (equivalently, the reduced norm of $\mathbb{H}^\times$) for each factor. As a representation, we can write

$$W \cong \bigotimes_{i=1}^{s} (\mathrm{Sym}^{k_i - 2} \mathbb{C}^2) \otimes \det{}^{l_i}$$

for $k_i \in \mathbb{Z}_{\geq 2}$ and $l_i \in \mathbb{C}$ for all $i \in [s]$. Because we require ω to be unitary, we must have $l_i = \frac{2-k_i}{2}$ – the square root makes sense here because the reduced norm of an element of $\mathbb{H}^\times$ is always positive. Because $\mathbb{H}^\times$ acts on functions in W by right translation, W is equipped with a left $\mathbb{H}^\times$-action. We write W_k for the space of functionals $f \colon W \to \mathbb{C}$ equipped with the right action of $h \in \mathbb{H}^\times$ given by $(f \cdot h)(\varphi') := f(h \cdot \varphi')$.

Definition 3.3. A quaternionic modular form of level $\mathcal{O}$ and nebentypus Ξ is a function

$$\tilde{\varphi} \colon B^\times \backslash B^\times(\mathbb{A}_F) \longrightarrow W_k$$

such that

(1) For any $\hat{u} \in \widehat{\mathcal{O}}^\times$, $\tilde{\varphi}(x\hat{u}) = \Xi(\hat{u})\tilde{\varphi}(x)$

(2) For any $\kappa \in K_{\infty,\mathbb{R}}$,

$$\tilde{\varphi}(x\kappa) = \frac{j(\kappa, \sqrt{-1})^k}{|\det \kappa|^{\frac{k}{2}}} \tilde{\varphi}(x). \tag{6}$$

(3) For any $h \in B_{\infty,\mathbb{C}}^\times$, $\tilde{\varphi}(xh) = \tilde{\varphi}(x)^h$.

We define

$$(T_{\mathfrak{p}}\tilde{\varphi})(x) := \sum_{j=1}^{P} \tilde{\varphi}(x\hat{\pi}_j^{-1})\Xi(\hat{\pi}_j). \tag{7}$$

Write ${}_h\varphi \in W$ for the right translate of $\varphi \in M_k^B(\mathcal{O},\Xi,\omega)_{\text{aut}}$ by $h \in (\mathbb{H}^\times)^s$.

Lemma 3.4. *The map*

$$\begin{aligned} F' \colon\ M_k^B(\mathcal{O},\Xi,\omega)_{\text{aut}} &\longrightarrow M_k^B(\mathcal{O},\Xi,\omega)'_{\text{aut}} \\ \varphi &\longmapsto (x \longmapsto ({}_h\varphi \longmapsto {}_h\varphi(x))) \end{aligned}$$

is a Hecke module isomorphism.

Proof. F' has an inverse map given by sending $x \in \mathbb{A}_F^\times$ to $(F'(\phi))(\phi)$ (the restriction of the functional $F'(\phi)$ to ϕ). The remaining properties are easy to check. □

Observe that $M_k^B(\mathcal{O},\Omega,\omega)_{\text{aut}}$ and $M_k^B(\mathcal{O},\Omega,\omega)'_{\text{aut}}$ are spaces of right K-equivariant and left $B^\times$-invariant functions. The point of introducing $M_k^B(\mathcal{O},\Omega,\omega)'_{\text{aut}}$ is that it helps us define a space of right K_∞-invariant functions.

Let $M_k^B(\mathcal{O},\Xi,\omega)^\pm$ denote the space of functions on $\mathcal{H}^{\pm r} \times \widehat{B}^\times \to W_k$ satisfying the conditions of Definition 3.1. By weak approximation [?, Proposition 28.7.3(b)], $B^\times$ has elements of all possible signs at the real places. It follows that the map from $M_k^B(\mathcal{O},\Xi,\omega)^\pm \to M_k^B(\mathcal{O},\Xi,\omega)$ given by restriction to $\mathcal{H}^r \times \widehat{B}^\times$ is a Hecke module isomorphism, so it suffices to prove that $M_k^B(\mathcal{O},\Omega,\omega)'_{\text{aut}} \cong M_k^B(\mathcal{O},\Omega,\omega)$.

In what follows, we write $x \in B^\times(\mathbb{A}_F)$ as $(x_\mathbb{R}, x_\mathbb{C}, \hat{x}) \in B^\times_{\infty,\mathbb{R}} \times B^\times_{\infty,\mathbb{C}} \times \widehat{B}^\times$.

Lemma 3.5. *The map F sending $\tilde{\varphi} \in M_k^B(\mathcal{O},\Xi,\omega)'_{\text{aut}}$ to*

$$\begin{aligned} F(\tilde{\varphi}) \colon\ B^\times_{\infty,\mathbb{R},+} \times \widehat{B}^\times &\longrightarrow W_k \\ (x_\mathbb{R}, \hat{x}) &\longmapsto \tfrac{(|\det x_\mathbb{R}|)^{k/2}}{j(x_\mathbb{R},\sqrt{-1})^k}\tilde{\varphi}(1, x_\mathbb{R}, \hat{x}) \end{aligned}$$

induces a Hecke module isomorphism $F\colon M_k(\mathcal{O},\Xi,\omega)'_{\text{aut}} \to M_k(\mathcal{O},\Xi,\omega)^\pm$.

Proof. By Equation (6), $\phi := F(\tilde{\varphi})$ is right $K_{\infty,\mathbb{R}}$-invariant, so we can think of ϕ as a function on

$$B^\times_{\infty,\mathbb{R}}/K_{\infty,\mathbb{R}} \times \widehat{B}^\times \cong \mathcal{H}^{\pm r} \times \widehat{B}^\times.$$

Furthermore,

$$\phi(\gamma(x_\mathbb{R},\hat{x}))^{\gamma_\mathbb{C}} = \phi(\gamma_\mathbb{R} x_\mathbb{R}, \hat{\gamma}\hat{x})^{\gamma_\mathbb{C}} = \frac{(|\det x_\mathbb{R}|)^{k/2}}{j(x_\mathbb{R},\sqrt{-1})^k}\tilde{\varphi}(\gamma(\gamma_\mathbb{C}^{-1}, x_\mathbb{R}, \hat{x}))^{\gamma_\mathbb{C}} = \frac{(|\det x_\mathbb{R}|)^{k/2}}{j(x_\mathbb{R},\sqrt{-1})^k}\tilde{\varphi}(1, x_\mathbb{R}, \hat{x}) = \phi(x_\mathbb{R},\hat{x})$$

and

$$\phi(x_\mathbb{R}, \hat{x}\hat{u}) = \tilde{\varphi}(1, x_\mathbb{R}, \hat{x}\hat{u}) = \Xi(\hat{u})\phi(x_\mathbb{R},\hat{x}).$$

Therefore, $\phi \in M_k^B(\mathcal{O},\Xi,\omega)^\pm$. Similarly, the map G sending $\phi \in M_k^B(\mathcal{O},\Xi,\omega)^\pm$ to

$$G(\phi)(x_\mathbb{C}, x_\mathbb{R}, \hat{x}) := \frac{j(x_\mathbb{R},\sqrt{-1})^k}{|\det x_\mathbb{R}|^{\frac{k}{2}}}\phi(x_\mathbb{R},\hat{x})^{x_\mathbb{C}}$$

has image in $M_k^B(\mathcal{O},\Xi,\omega)'_{\text{aut}}$, and we can check that it is the inverse of F. The map F is compatible with the Hecke actions in Equation (4) and Equation (5), so it is a Hecke module isomorphism. □

4. Preliminaries on computing quaternionic modular forms with nebentypus

One issue with the description of Definition 3.1 is that it is not immediately clear how to produce an explicit basis for the space. Even given a basis, the formula of Equation (5) involves multiplication in the adèles, which is computationally inconvenient. In this section, we will produce a decomposition of $S_k^B(\mathcal{O},\Xi)$ (Lemma 4.2) and an alternative description of the Hecke action on $S_k^B(\mathcal{O},\Xi)$ in terms of this decomposition (Lemma 4.3). In particular, the formula of Lemma 4.3 will be entirely in terms of elements of $B^\times$, rather than elements of $B^\times(\mathbb{A}_F)$.

4.1. **Components of the quaternionic Shimura variety.** While the forms in $M_k^B(\mathcal{O},\Xi)$ (as in Definition 3.1) are not quite right $\widehat{\mathcal{O}}^\times$-invariant when Ξ is nontrivial, they are determined by their values on representatives in

$$Y^B(\mathcal{O}) := B_+^\times \backslash \left(\mathcal{H}^r \times \widehat{B}^\times / \widehat{\mathcal{O}}^\times\right). \tag{8}$$

We call $Y^B(\mathcal{O})$ the quaternionic Shimura variety associated to $\mathcal{O}$.

Remark. When $B \not\cong M_{2,F}$ $Y^B(\mathcal{O})$ has no cusps, and $Y^B(\mathcal{O}) = X_B(\mathcal{O})$ is compact.

Because $\mathcal{H}^r$ is connected, the connected components of $Y^B(\mathcal{O})$ correspond to the elements of $B_+^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times$.

Theorem 4.1 ([**?**, 28.4.3, 27.7.1, 28.5.5])**.**

(1) The map

$$\begin{aligned} F\colon\ B_+^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times &\longrightarrow \operatorname{Cls}\mathcal{O} \\ B_+^\times \hat{\alpha}\widehat{\mathcal{O}}^\times &\longmapsto \hat{\alpha}\widehat{\mathcal{O}} \cap B \end{aligned}$$

is a bijection. In particular, $Y^B(\mathcal{O})$ has finitely many connected components.

(2) The map

$$\operatorname{nrd}\colon B_+^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times \longrightarrow F_+^\times \backslash \widehat{F}^\times / \operatorname{nrd}(\widehat{\mathcal{O}}^\times) \tag{9}$$

is a surjection, and if B is indefinite, then it is a bijection.

(3) When $\mathcal{O}$ is a maximal or Eichler order, $\operatorname{nrd}(\widehat{\mathcal{O}}^\times) = \widehat{\mathbb{Z}_F}^\times$ and $F_+^\times \backslash \widehat{F}^\times / \operatorname{nrd}(\widehat{\mathcal{O}}^\times) \cong \operatorname{Cl}_F^+$.

Let H be the the cardinality of

$$B_+^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times$$

, and pick representatives $\{\hat{\alpha}_h\}_{h\in[H]}$. Given $\phi \in M_k^B(\mathcal{O},\Xi)$ and $h \in [H]$, we can define a function

$$\begin{aligned} \phi_h\colon\ \mathcal{H}^r &\longrightarrow W_k \\ z &\longmapsto \phi(z, \hat{\alpha}_h). \end{aligned} \tag{10}$$

For any function $\phi_h \colon \mathcal{H}^r \to W_k$, we define an action

$$\phi_h|_k\gamma := \frac{(\det\gamma)^{k/2}}{j(\gamma,z)^k}\phi_h(\gamma z)^\gamma. \tag{11}$$

Let $\mathcal{O}_h := \hat{\alpha}_h\widehat{\mathcal{O}}\hat{\alpha}_h^{-1} \cap B$, and define

$$M_k^B(\mathcal{O},\Xi;h) := \ \{\phi_h \colon \mathcal{H}^r \longrightarrow W_k \colon\ \phi_h|_k\gamma = \Xi(\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha_h})\phi_h \ \text{ for } \gamma \in \mathcal{O}_h^\times\}. \tag{12}$$

Note that $\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha_h} \in \widehat{\mathcal{O}}^\times$, so it makes sense to evaluate Ξ on it.

Lemma 4.2. *The map*

$$\begin{aligned} \Phi\colon\ M_k^B(\mathcal{O},\Xi) &\longrightarrow \textstyle\bigoplus_{h=1}^H M_k^B(\mathcal{O},\Xi;h) \\ \phi &\longmapsto (\phi_h)_{h\in[H]} \end{aligned}$$

is an isomorphism.

Proof. The map Φ is well-defined because for $\gamma \in \mathcal{O}_h^\times$,

$$(\phi_h|_k\gamma)(z) \ = \frac{(\det\gamma)^{k/2}}{j(\gamma,z)^k}\phi(\gamma z, \hat{\alpha}_h)^\gamma \ = \phi(z, \gamma^{-1}\hat{\alpha}_h) \ = \phi(z, \hat{\alpha}_h(\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h)) \ = \Xi(\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h)\phi_h(z).$$

It is an isomorphism because by Definition 3.1, knowing $\phi(z,\hat{\alpha}_h)$ for all $z \in \mathcal{H}^r$ and $h \in [H]$ is enough to recover ϕ on all of $\mathcal{H}^r \times \widehat{B}^\times$. □

Recall the definition of Hecke operators on $M_k^B(\mathcal{O},\Xi)$ in Equation (5). We can ask how the Hecke operator interacts with the isomorphism of Lemma 4.2.

Lemma 4.3. *There exist:*

(1) A function $j^*\colon [H] \to [H]$ for every $j \in [P]$;
(2) Elements

$$\left\{\varpi_{j,h} \in \hat{\alpha}_{j^*(h)}\hat{\mathcal{O}}^\times\hat{\pi}_j\hat{\alpha}_h^{-1} \ \cap B_+^\times\right\}_{\substack{j\in[P]\\ h\in[H]}},$$

where each $\varpi_{j,h}$ is well-defined up to multiplication on the left by $\mathcal{O}_{j^*(h)}^\times$;

such that

$$(T_\mathfrak{p}\phi)_h(z) = \sum_{j=1}^{P} \Xi(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)(\phi_{j^*(h)}|_k\varpi_{j,h})(z). \tag{13}$$

Proof. By strong approximation (Theorem 4.1), for all $h \in [H]$ and $j \in [P]$, there exist $\varpi_{j,h} \in B_+^\times$, $j^*(h) \in [H]$, and $\hat{u} \in \widehat{\mathcal{O}}^\times$ such that $\hat{\alpha}_h\hat{\pi}_j^{-1} = \varpi_{j,h}^{-1}\hat{\alpha}_{j^*(h)}\hat{u}$. Applying this to Equation (5),

$$\begin{aligned}(T_\mathfrak{p}\phi)_h(z) &= \sum_{j=1}^{P} \phi(z, \varpi_{j,h}^{-1}\hat{\alpha}_{j^*(h)}\hat{u})\Xi(\hat{\pi}_j)\\ &= \sum_{j=1}^{P} \Xi(\hat{u}\hat{\pi}_j)(\phi_{j^*(h)}|_k\varpi_{j,h})(z)\\ &= \sum_{j=1}^{P} \Xi(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)(\phi_{j^*(h)}|_k\varpi_{j,h})(z). \end{aligned}\tag{14}$$

If for some j and h we have $\varpi_{j,h}^{-1}\hat{\alpha}_{j^*(h)}\hat{u} = (\varpi'_{j,h})^{-1}\hat{\alpha}_{j^*(h)}\hat{u}'$, then $\varpi'_{j,h} = \gamma\varpi_{j,h}$ for some $\gamma \in \mathcal{O}_{j^*(h)}^\times$. By Equation (12), replacing $\varpi_{j,h}$ with $\varpi'_{j,h} = \gamma\varpi_{j,h}$ does not change the summand in Equation (14). □

Here is one way to think about the sets $\{\varpi_{j,h}\}_{j\in[P]}$ for each $h \in [H]$. By definition,

$$\varpi_{j,h} \in (\hat{\alpha}_{j(h)}\widehat{\mathcal{O}}^\times\hat{\alpha}_{j(h)}^{-1})\backslash\hat{\alpha}_{j(h)}\widehat{\mathcal{O}}^\times\hat{\pi}_j\hat{\alpha}_h^{-1}.$$

Applying Equation (3), we see that for a fixed h, the elements $\{\alpha_{j(h)}^{-1}\varpi_{j,h}\}_{j\in[P]}$ are in

$$\widehat{\mathcal{O}}^\times\backslash\widehat{\mathcal{O}}^\times\hat{\pi}\widehat{\mathcal{O}}^\times\hat{\alpha}_h^{-1} = \sqcup_{j=1}^{P}\widehat{\mathcal{O}}^\times\hat{\pi}_j\hat{\alpha}_h^{-1}.$$

By definition of the $\{\hat{\alpha}_h\}$ and Theorem 4.1, for each $j \in [P]$, there is a unique $\hat{\alpha}_{j(h)}$ for which $\hat{\alpha}_{j(h)}\widehat{\mathcal{O}}^\times\hat{\pi}_j\hat{\alpha}_h^{-1} \cap B_+^\times \neq \emptyset$. Said differently, we can construct $\{\varpi_{j,h}\}_{j\in[P]}$ by considering the quotients

$$(\hat{\alpha}_{h'}\widehat{\mathcal{O}}^\times\hat{\alpha}_{h'}^{-1})\backslash\hat{\alpha}_{h'}\widehat{\mathcal{O}}^\times\hat{\pi}\widehat{\mathcal{O}}^\times\hat{\alpha}_h^{-1} = \sqcup_{j=1}^{P}(\hat{\alpha}_{h'}\widehat{\mathcal{O}}^\times\hat{\alpha}_{h'}^{-1})\backslash\hat{\alpha}_{h'}\widehat{\mathcal{O}}^\times\hat{\pi}_j\hat{\alpha}_h^{-1}$$

for all $h' \in [H]$. For each $j \in [P]$, there will be a unique $h' =: j(h)$ for which the corresponding coset has nonempty intersection with $B_+^\times$, and we take $\varpi_{j,h}$ to be any $B_+^\times$ representative of this (unique) coset.

Remark. If $h = 1$, then the quotient of interest becomes $\widehat{\mathcal{O}}^\times\backslash\widehat{\mathcal{O}}^\times\hat{\pi}\widehat{\mathcal{O}}^\times$, and we are back in the setting of Equation (4) – in this case, we can always choose $\hat{\pi}_j^{-1} = \varpi_{j,h} \in B_+^\times$, and the machinery of Lemma 4.3 is unnecessary. The complication is entirely to deal with the fact that the quaternionic Shimura variety can have many different components.

Lemma 4.4. *Fix $h \in [H]$. The set of cosets $\{\mathcal{O}_{j(h)}^\times\varpi_{j,h}\}_{j\in[P]}$ is invariant under multiplication on the right by $\gamma \in \mathcal{O}_h^\times$. Furthermore, if $\gamma(j) \in [P]$ is such that $\varpi_{j,h}\gamma = \mathcal{O}_{\gamma(j)(h)}^\times\varpi_{\gamma(j),h}$, then $\gamma(j)(h) = j(h)$.*

Proof. Write $\gamma = \hat{\alpha}_h\hat{u}_\gamma\hat{\alpha}_h^{-1}$ for $\hat{u}_\gamma \in \widehat{\mathcal{O}}^\times$. Then, $\varpi_{j,h}\gamma \in \hat{\alpha}_{j(h)}\widehat{\mathcal{O}}^\times\hat{\pi}_j^{-1}\hat{u}_\gamma\hat{\alpha}_h^{-1}$. Multiplication by an element of $\widehat{\mathcal{O}}^\times$ permutes the cosets $\sqcup\widehat{\mathcal{O}}^\times\hat{\pi}_j^{-1}$, so the result follows. □

[**?**, Equation 7.24] describes, in the case of trivial nebentypus, how we can avoid doing computation with adèles by replacing the double coset representatives $\{\hat{\alpha}_h\}_{h\in[H]}$ with the corresponding right $\mathcal{O}$-ideals $\{I_h := \hat{\alpha}_h\hat{\mathcal{O}} \cap B\}$ (representatives of the corresponding right ideal classes in $\mathcal{O}$) and computing the $\varpi_{j,h}$ in terms of these ideals. Observe that by Lemma 4.4, for any given $h \in [H]$, the cosets $\{\mathcal{O}_{j(h)}^\times\varpi_{j,h}\}_{j\in[P]}$ depend only on the cosets $\{\hat{\alpha}_h\widehat{\mathcal{O}}^\times\}$, and hence only on the ideals I_h.

We would like to evaluate the expression $\Xi(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)$ in Equation (13) without working adèlically. By weak approximation [?, Proposition 28.7.3(b)], we may choose $\{\hat{\alpha}_h\}$ such that $\Omega(\hat{\alpha}_h) = 1$ for all $h \in [H]$, since the kernel of Ω is an open subgroup of $\widehat{B}^\times$ that can be smaller than $\mathcal{O}_\mathfrak{p}$ only for $\mathfrak{p} \mid \mathfrak{N}$. For such choices of $\{\hat{\alpha}_h\}$, $\varpi_{j,h} \in \hat{\alpha}_{j^*(h)}\hat{u}\hat{\pi}_j\hat{\alpha}_h^{-1}$ lies in $\mathcal{O}_\mathfrak{p}$ for every $\mathfrak{p} \mid \mathfrak{N}$, and we have

$$\Xi(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h) = \Xi(\hat{\alpha}_{j^*(h)}^{-1})\Xi(\varpi_{j,h})\Xi(\hat{\alpha}_h).$$

We may worry that our actual computation involves the ideals $\{I_h\}$ rather than the adèlic elements $\{\hat{\alpha}_h\}$ – while we can still require that I_h be coprime to the conductor of Ω, the condition that $\Omega(\hat{\alpha}_h) = 1$ is not detected by the coset $\hat{\alpha}_h\widehat{\mathcal{O}}^\times$ and hence cannot be expressed in terms of the $\{I_h\}$. There are a few ways around this, but here is one.

With $\{\hat{\alpha}_h\}$ chosen so that the $\{I_h\}$ are coprime to the conductor of Ω, let $T_\mathfrak{p}$ be the matrix of the Hecke operator on $\bigoplus_h M_k^B(\mathcal{O}, \Xi; h)$ defined by Equation (13), and let $T'_\mathfrak{p}$ be the matrix defined by Equation (13) after replacing $\Xi(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)$ with $\Xi(\varpi_{j,h})$. Let

$$D := \operatorname{diag}\left(\ldots, \underbrace{\Xi(\hat{\alpha}_h), \ldots, \Xi(\hat{\alpha}_h)}_{\dim M_k^B(\mathcal{O},\Xi;h) \text{ times}}, \ldots\right).$$

Then, $T_\mathfrak{p} = DT'_\mathfrak{p}D^{-1}$. Since D is independent of $\mathfrak{p}$, the $\{T'_\mathfrak{p}\}$ and $\{T_\mathfrak{p}\}$ differ only by a change of basis. Therefore, we lose nothing by computing $\{T'_\mathfrak{p}\}$ – avoiding any adelic computation – once we have chosen $\{\hat{\alpha}_h\}$ appropriately.

4.2. **Why Lemma 4.3 is not enough.** Lemma 4.3 is not quite computationally useful. When $r = 1$, the spaces $M_k^B(\mathcal{O}, \Xi; h)$ in Equation (12) are spaces of holomorphic functions on $\mathcal{H}$ satisfying conditions similar to those defining a modular form. The Hecke action in Lemma 4.3 generalizes the classical formula

$$T_p f = \sum_j f|_k \omega_j,$$

for the Hecke operator on a modular form. This is not ideal for computational purposes – we would like a description involving only discrete quantities and linear algebra, with no analog objects in the mix.

When $r = 0$, the quaternionic Shimura variety is a disjoint union of points, and each $M_k(\mathcal{O}, \Xi; h)$ can be specified by an element of W_k. Then, Lemma 4.3 actually gives us a usable formula for computing the Hecke action on this space. However, this procedure is inefficient when computing spaces of of many different levels $\mathcal{O}$ (e.g. when computing tables of Hilbert modular forms) as for each $\mathcal{O}$ we need new sets of representatives $\{\hat{\alpha}_h\}$ and elements $\{\varpi_{j,h}\}$.

In the following sections, we rectify these issues. In particular, we describe how to compute spaces with varying $\mathcal{O} = \mathcal{O}_0(\mathfrak{N})$ using only the $\{\hat{\alpha}_h\}$ and $\{\varpi_{j,h}\}$ associated to $\mathcal{O} = \mathcal{O}_0(1)$ (or slight modifications thereof).

5. Indefinite method

In this section, we will describe a procedure for computing $S_k^B(\mathcal{O}, \Xi)$ when $r = 1$, i.e. when B is split at exactly one infinite place. Algorithms to compute $M_k^B(\mathcal{O}, 1)$ for such B were developed by [?]. They were extended to fields with arbitrary narrow class number by [?]. In this section, we explain how to extend their methods to arbitrary nebentypus Ξ. The idea of these approaches is to use a generalization of the Eichler-Shimura isomorphism (Theorem 5.2) to realize each $S_k^B(\mathcal{O}, \Xi; h)$ from Lemma 4.2 within the first cohomology of an arithmetic group (depending on the level $\mathcal{O}$ and on h) with some coefficients (depending on the weight k and nebentypus Ξ). We start with an example of the method applied to weight 2 modular forms with nebentypus.

Example 5.1. Let $B = M_{2,\mathbb{Q}}$ and $\mathcal{O} \subseteq M_2(\mathbb{Z})$ be an order. For simplicity, assume that $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \in \mathcal{O}$. Let $\tilde{\Gamma} := \mathcal{O}^\times / \pm 1 \subset \mathrm{PGL}_2(\mathbb{Z})$ and $\Gamma := \mathcal{O}^1 / \pm 1 \subset \mathrm{PSL}_2(\mathbb{Z})$, where $\mathcal{O}^1$ denotes the elements of $\mathcal{O}^\times$ with determinant 1. Given $f \in S_2(\Gamma, \chi)$, the differential $\omega_f := f(z)dz$ defines a closed holomorphic differential on $\mathcal{H}$. It is not quite Γ-invariant because χ is nontrivial, but writing $X := \Gamma \backslash \mathcal{H}$, one can check that

$\omega_f \in H^0(X, \mathbb{C}(\chi) \otimes_{\mathbb{C}} \Omega^1_X)$, where $\mathbb{C}(\chi)$ is the local system associated to the representation χ of $\Gamma = \pi_1(X)$. We claim that there is a $\mathbb{C}$-linear Hecke module isomorphism

$$S_2(\Gamma, \chi) \cong H^1(\Gamma, \mathbb{C}(\chi))^{\tilde{\Gamma}/\Gamma}, \tag{15}$$

where for $\mu \in \tilde{\Gamma}$, the (left) action of μ on $\xi \in H^1(\Gamma, \mathbb{C}(\chi))$ is given by $(\mu \bullet \xi)(\gamma) = \mu \cdot \xi(\mu\gamma\mu^{-1})$. Given a base point $P \in X$, the integral

$$\begin{aligned} \xi_f \colon \Gamma &\longrightarrow \mathbb{C}(\chi) \\ \gamma &\longmapsto \textstyle\int_P^{\gamma P} \omega_f(z) \end{aligned}$$

is well-defined (it depends only on the homotopy class of the path $P \sim \gamma P$ in $\mathcal{H}$). The map ξ_f is in fact a 1-cocycle, as

$$\int_P^{\gamma\gamma' P} \omega_f \ = \xi_f(\gamma) + \int_P^{\gamma' P} \gamma^*\omega_f \ = \xi_f(\gamma) + \chi_0(\gamma) \int_P^{\gamma' P} \omega_f \ = \xi_f(\gamma) + \gamma \cdot \xi_f(\gamma').$$

Therefore, we have a map of vector spaces

$$F \colon S_2(\Gamma) \xrightarrow{\sim} H^0(X, \mathbb{C}(\chi)) \longrightarrow H^1(\Gamma, \mathbb{C})$$

taking f to ξ_f, and we can check that F is an injection. By Lemma 5.5,

$$H^1(\Gamma, \mathbb{C}(\chi_0)) \ \cong H^0(X, \mathbb{C}(\chi) \otimes_{\underline{\mathbb{C}}_X} \Omega^1_X) \ \oplus \overline{H^1(X, \mathbb{C}(\chi) \otimes_{\underline{\mathbb{C}}_X} \mathcal{O}_X)} \ \cong S_2(\Gamma, \chi) \oplus \overline{S_2(\Gamma, \chi^{-1})}. \tag{16}$$

Take $\mu = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \in \tilde{\Gamma} \setminus \Gamma$ – because $\tilde{\Gamma}/\Gamma \cong \mathbb{Z}/2\mathbb{Z}$, any choice of μ will work, but this choice is convenient. Then,

$$\int_P^{\mu\gamma\mu^{-1} P} f(z) dz = \int_{\mu P}^{\mu\gamma P} f(z) dz = \int_P^{\gamma P} f(\mu z) d(\mu z) = \int_P^{\gamma P} -f(-\overline{z}) d\overline{z}.$$

As such, $\mu \bullet h_f$ is antiholomorphic. We can check that if $f(z) = \sum_m a_m \exp(2\pi\sqrt{-1}mz)$, then $\overline{f(-\overline{z})}(z) = \sum_m \overline{a_m} \exp(2\pi\sqrt{-1}mz) =: f^c(z)$. In particular, it follows that the action of μ commutes with the Hecke action, and so we have a sequence of Hecke module isomorphisms

$$S_2(\Gamma, \chi) \xrightarrow{F} H^1(\Gamma, \mathbb{C}(\chi_0)) \xrightarrow{\xi \longmapsto (\xi + \mu \bullet \xi)} H^1(\Gamma, \mathbb{C}(\chi_0))^{\tilde{\Gamma}/\Gamma}.$$

In the main result of this section, Theorem 5.2, we replace Γ with an arbitrary arithmetic Fuchsian group over a totally real field, and allow higher weight. Nonetheless, the idea of the proof is exactly the same as that of Example 5.1.

5.1. **Defining the coefficients of the cohomology.** The representation $\mathrm{Sym}^m \mathbb{C}^2$ can be described as the space of homogeneous bivariate degree m polynomials $\mathbb{C}[X, Y]_m$ (which we think of as functions on the column vector $(X\ Y)^T$) with a right action of $\mathrm{GL}_2(\mathbb{C})$ given by precomposing with $\gamma \in \mathrm{GL}_2(\mathbb{C})$. We will instead want to think of this as a space with a left[2] action of $\gamma \in \mathrm{GL}_2(\mathbb{C})$ given by precomposing with the transpose γ^T, i.e. $\gamma \cdot f(X, Y) = f(aX + bY, cX + dY)$.

Let k_1 denote the component of the weight at the (unique) infinite place of F split in B. Then,

$$V_k := \Big((\mathrm{Sym}^{k_1 - 2} \mathbb{C}^2) \otimes \det^{2 - k_1/2} \Big) \otimes W_k \cong \bigotimes_{i \in [n]} \mathrm{Sym}^{k_i - 2} \mathbb{C}^2 \otimes \det^{(\underline{k_0} - k_i)/2}, \tag{17}$$

where the isomorphism follows from the self-duality of $(\mathrm{Sym}^{k_i - 2} \mathbb{C}^2) \otimes \det^{2 - k_i/2}$. We equip this space with a left action of $B^\times$ whereby given $\gamma \in B^\times$ and $v_1 \otimes w \otimes \in \mathbb{C}[X, Y]_{k_{\mathbb{R}}} \otimes W_k$, we define

$$\gamma \cdot (v_1 \otimes w) = (\gamma \cdot v_1) \otimes w^{\gamma^{-1}} = v \circ \gamma^T \otimes w^{\gamma^{-1}}.$$

We define $\Xi_h \colon \widehat{\mathcal{O}}_h \to \mathbb{C}$ by $\Xi_h(\hat{x}) := \Xi(\hat{\alpha}_h^{-1} \hat{x} \hat{\alpha}_h)$. We define $V_{k, \Xi_h} := V_k \otimes \Xi_h^{-1}$.

By the Hasse-Schilling theorem (see [?, Main Theorem 14.7.4]) (or simply by noting that the reduced norm on $\mathbb{H}^\times$ is always positive), for any $\gamma \in B^\times$, $\mathrm{nrd}(\gamma)$ is positive at every infinite place of F which is ramified in B.

[2] The convention in the literature is to give V_k a right action. However, I felt that this would make the proof of Theorem 5.2 less readable.

Since $r = 1$, this means that $\mathrm{nrd}(\gamma)$ is positive at all but one infinite place. Let $\mathcal{O}_h^1 := \{\gamma \in \mathcal{O}_h^\times : \mathrm{nrd}(\gamma) = 1\}$, $\tilde{\Gamma}_h := \mathcal{O}_h^\times/(\mathbb{Z}_F^\times \cap \mathcal{O}_h^\times)$, and $\Gamma_h := \mathcal{O}_h^1/(\mathbb{Z}_F^\times \cap \mathcal{O}_h^1)$. Because $\mathcal{O}_h^\times$ contains an element of negative norm, there is an exact sequence

$$1 \longrightarrow \Gamma_h \longrightarrow \tilde{\Gamma}_h \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \tag{18}$$

Because V_{k,Ξ_h} is trivial on scalars, it descends to a representation of $B^\times/F^\times$.

5.2. **The Eichler-Shimura isomorphism.** With this in hand, we now prove the generalization of Equation (15) that we need.

5.3. **Proving a vector space isomorphism.**

Theorem 5.2. *Let B/F be split at exactly one infinite place. Then, there is a $\mathbb{C}$-linear isomorphism*

$$S_k^B(\mathcal{O}, \Xi; h) \cong H^1(\Gamma_h, V_{k,\Xi_h})^{\tilde{\Gamma}_h/\Gamma_h}, \tag{19}$$

and in particular

$$S_k^B(\mathcal{O}, \Xi) \cong \bigoplus_{h \in [h^+(F)]} H^1(\Gamma_h, V_{k,\Xi_h})^{\tilde{\Gamma}_h/\Gamma_h}.$$

For the remainder of this subsection, we prove Theorem 5.2. Because we prove the isomorphism of Equation (19) for each $h \in [H]$ separately, we may simplify our notation by fixing $V_{k,\Xi} := V_{k,\Xi_h}$, $\tilde{\Gamma} := \tilde{\Gamma}_h$, $\Gamma := \Gamma_h$, and $X := X_h := \Gamma_h \backslash \mathcal{H}$.

Given $f \in S_k(\mathcal{O}, \Xi; h)$, we associate to f a $V_{k,\Xi} = \mathbb{C}[X,Y]_{(k_1-2)} \otimes W_k$-valued differential 1-form on $\mathcal{H}$:

$$\omega_f(z) := f(z)(zX + Y)^{k-2} dz.$$

This differential is holomorphic and closed. Rather than directly showing it is Γ-invariant (and hence that it descends to $X(\Gamma_h)$, we prove a slightly more general fact which will be useful later when studying the Hecke action. Given $\beta \in B^\times$ and a $V_{k,\Xi}$-valued differential form ω on $\mathcal{H}$, we write $\beta \cdot \omega$ for the form produced by postcomposing ω with the left action of β on $V_{k,\Xi}$.

Lemma 5.3. *Let $\beta \in B_+^\times$. Then,*

$$\beta^* \omega_{(\Xi(\beta^{-1} f|_k \beta^{-1})} = \beta \cdot \omega_f.$$

Proof. For $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{aligned} \omega_{f|_k \beta^{-1}}(\beta z) &= \Xi(\beta^{-1})(\det \beta)^{2-k/2}(cz+d)^{k-2} f(z)(\frac{az+b}{cz+d}X + Y)^{k-2} dz \\ &= \Xi(\beta^{-1})(\det \beta)^{2-k/2} f(z)((aX+cY)z + (bX+dY))^{k-2} dz \\ &= \beta \cdot \omega_f. \end{aligned}$$

□

It follows that for $\gamma \in \mathcal{O}_h^1$,

$$\gamma^* \omega_f = \gamma \cdot \omega_f, \tag{20}$$

so in particular ω_f descends to a $V_{k,\Xi}$-valued differential form on X. We then have

$$\int_{\gamma A}^{\gamma B} \omega_f = \gamma \cdot \int_A^B \omega_f. \tag{21}$$

Abusing notation slightly, we denote by $V_{k,\Xi}$ the local system on X corresponding to the monodromy representation $V_{k,\Xi}$, and write $(\mathcal{V}_{k,\Xi}, \nabla)$ (where $\mathcal{V}_{k,\Xi} := V_{k,\Xi} \otimes_{\underline{\mathbb{C}}_X} \mathcal{O}_X$) for the (holomorphic) vector bundle with flat connection whose flat sections are $V_{k,\Xi}$. Then, $\omega_f \in H^0(X, \mathcal{V}_{k,\Xi} \otimes \Omega_X^1)$ by definition.

Lemma 5.4. *The map sending f to ω_f induces a $\mathbb{C}$-linear isomorphism from $S_k^B(\mathcal{O}, \Xi; h)$ to $H^0(X, \mathcal{V}_{k,\Xi} \otimes \Omega_X^1)$.*

Sketch. The form ω_f is zero if and only if f is. We leave the proof of surjectivity as a nontrivial exercise. One approach is to note that any element of $H^0(X, \mathcal{V}_{k,\Xi} \otimes \Omega^1_X)$ can be expressed as $P(X,Y;z)dz$ where

$$\Xi(\gamma)(cz+d)^2 P(aX+cY, bX+dY; z) = P(X,Y; \frac{ax+b}{cz+d})$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. From this, the result can be deduced with some effort. □

Lemma 5.5.

$$H^1(\Gamma, V_{k,\xi}) \cong S_k^B(\mathcal{O}, \Xi; h) \oplus \overline{S_k^B(\mathcal{O}, \Xi^{-1}; h)}. \tag{22}$$

This should look like the usual statement of the Eichler-Shimura isomorphism with nebentypus [**?**, 6.3].

Proof. Building the Hodge-de Rham spectral sequence from the resolution

$$0 \longrightarrow V_{k,\Xi} \hookrightarrow \mathcal{V}_k \xrightarrow{\nabla} \mathcal{V}_k \otimes \Omega^1 \xrightarrow{\nabla} \mathcal{V}_k \otimes \Omega^2 \xrightarrow{\nabla} \cdots$$

of $V_{k,\Xi}$, we obtain a Hodge decomposition

$$H^1(\Gamma, V_{k,\Xi}) \cong H^0(X, \mathcal{V}_{k,\Xi} \otimes \Omega^1_X) \oplus H^1(X, \mathcal{V}_{k,\Xi} \otimes \mathcal{O}_X).$$

By Serre duality,

$$H^1(X, \mathcal{V}_{k,\Xi} \otimes \mathcal{O}_X) \cong H^0(X, \mathcal{V}^*_{k,\Xi} \otimes \Omega^1_X)^*.$$

Since $\mathcal{V}_{k,\Xi}$ is the vector bundle associated to the representation $V_{k,\Xi}$, one can check that $\mathcal{V}^*_{k,\Xi}$ is the vector bundle associated to the dual representation $V^*_{k,\Xi} \cong V_{k,\Xi^{-1}}$.

By the Hodge index theorem, which goes through in the same way with local system coefficients, there is a Hermitian pairing on $H^1(\Gamma_h, V^*_{k,\Xi})$ which restricts to a Hermitian pairing on $H^0(X, \mathcal{V}^*_{k,\Xi} \otimes \Omega^1_X)$. We deduce

$$H^0(X, \mathcal{V}^*_{k,\Xi} \otimes \Omega^1_X)^* \cong \overline{H^0(X, \mathcal{V}^*_{k,\Xi} \otimes \Omega^1_X)} \cong \overline{H^0(X, \mathcal{V}_{k,\Xi^{-1}} \otimes \Omega^1_X)}.$$

The result then follows from Lemma 5.4. □

Pick an arbitrary basepoint $P \in \mathcal{H}$. To any $f \in S_k^B(\mathcal{O}, \Xi; h)$, we can associate a function

$$\begin{aligned} \xi_f \colon \Gamma_h &\longrightarrow W_k \\ \gamma &\longmapsto \int_{P_h}^{\gamma P_h} f(z)(zX+Y)^{k-2} dz \end{aligned}$$

where $i = (i_j)_{j \in [n]}$ is such that $i_j \leq k_j - 2$ for all i. The function ξ_f is a 1-cocycle, as

$$\int_P^{\gamma\gamma' P} \omega_f = \int_P^{\gamma P} \omega_f + \int_{\gamma P}^{\gamma\gamma' P} \omega_f \ = \xi_f(\gamma) + \gamma \cdot \xi_f(\gamma').$$

Because ω_f is closed, this integral only depends on the path from P to γP up to homotopy. The class of ξ_f in $H^1(\Gamma, V_k)$ is independent of the choice of base point P. As such, we have a well-defined composition $F \colon S_k^B(\mathcal{O}, \Xi) \xrightarrow{f \mapsto \omega_f} H^0(X, \mathcal{V}_{k,\Xi} \otimes \Omega^1_X) \to H^1(\Gamma, V_{k,\Xi})$ sending f to ξ_f. The first map is an isomorphism by Lemma 5.4, and the second is an injection because any $((V_{k,\Xi})$-valued) differential form on X is determined by its $((V_{k,\Xi})$-valued)-integrals on a homology basis for X.

We now extend the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathcal{H}$ to an action of $\mathrm{PGL}_2(\mathbb{R})$ by defining, for $\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \setminus \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathcal{H}$,

$$\mu z := \frac{a\overline{z} + b}{c\overline{z} + d}.$$

Write $Z := \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$.

Lemma 5.6. *For $f \in S_k^B(\mathcal{O}, \Xi; h)$ and $\mu \in \tilde{\Gamma} \setminus \Gamma$,*

$$\mu^{-1} \cdot (\mu^* \omega_f) = \overline{\omega_{f^c}}$$

for $f^c := f(-\overline{z})^Z \in S_k^B(\mathcal{O}, \Xi^{-1}; h)$.

Proof.

$$\begin{aligned}\mu^{-1}\cdot(\mu^*\omega_f(z)) &= \mu^{-1}\cdot\left(f(\mu z)\left(\frac{a\overline{z}+b}{c\overline{z}+d}X+Y\right)^{k-2}d(\mu z)\right)\\ &= f((\mu Z^{-1})Zz)^{\mu}(c\overline{z}+d)^{2-k}(\overline{z}X+Y)^{k-2}d(\mu z)\\ &= f(-\overline{z})^Z(\overline{z}X+Y)^{k-2}d\overline{z}.\end{aligned}$$

This is certainly an antiholomorphic differential. We want to show that $f^c(z) := f(-\overline{z})^Z$ is in $S_k^B(\mathcal{O},\Xi^{-1};h)$. For $\gamma\in\Gamma_h$,

$$(f^c|_k\gamma)(z) = \frac{(\det\gamma)^{k/2}}{(cz+d)^k}\left(\overline{f((Z\gamma Z^{-1})Zz)^Z}\right)^{\gamma} = \frac{(\det\gamma)^{k/2}}{(cz+d)^k}\left(\overline{\Xi(\gamma^{-1}\frac{(c\overline{z}+d)^k}{(\det\gamma)^{k/2}}f(Zz)^{Z\gamma^{-1}}}\right) = f^c(z)\Xi^{-1}(\gamma^{-1}).$$

□

Corollary 5.7. *For $f\in S_k^B(\mathcal{O},\Xi;h)$ and $\mu\in\tilde{\Gamma}\setminus\Gamma$,*

$$\mu\bullet\xi_f = \overline{\xi_{f^c}}.$$

Proof. We have

$$\mu^{-1}\cdot\int_P^{\mu\gamma\mu^{-1}P}\omega_f = \int_{\mu^{-1}P}^{\gamma\mu^{-1}P}\mu^{-1}\cdot\mu^*\omega_f = \int_{\mu^{-1}P}^{\gamma\mu^{-1}P}\overline{\omega_{f^c}}.$$

Because changing the basepoint from $\mu^{-1}P$ to P does not affect the cohomology class of ξ_g, the result follows. □

It follows that the μ action on $H^1(X,V_{k,\Xi})$ interchanges the pieces of the Hodge decomposition in Equation (22). Writing $H^1(X,V_{k,\Xi})^{\mu=1}$ for the fixed points under the μ-action, the dimension of $H^1(X,V_{k,\Xi})^{\mu=1}$ is exactly half that of $H^1(X,V_{k,\Xi})$, and in particular

$$\dim S_k^B(\mathcal{O},\Xi;h) = \dim H^0(X,\mathcal{V}_{k,\Xi}\otimes\Omega_X^1) = \frac{1}{2}\dim H^1(X,V_{k,\Xi}) = \dim H^1(\Gamma_h,V_{k,\Xi})^{\mu=1}.$$

We can realize the isomorphism by the map

$$S_k^B(\mathcal{O},\Xi)\xrightarrow{f\longmapsto\omega_f}H^0(X,\mathcal{V}_{k,\Xi}\otimes\Omega_X^1)\xrightarrow{\omega_f\longmapsto\xi_f}H^1(\Gamma,V_{k,\Xi})\xrightarrow{\xi_f\longmapsto(\xi_f+\mu\bullet\xi_f)}H^1(\Gamma,V_{k,\Xi})^{\mu=1} \tag{23}$$

This concludes the proof of Theorem 5.2.

5.4. **Proving a Hecke module isomorphism.** We now proceed to describe the Hecke action on $\bigoplus_h H^1(\Gamma_h,V_{k,\Xi_h})$. The Hecke operators will interlace the data from the various direct summands, so our notation will need to keep track of this – as such, we reintroduce the subscripts $h\in[H]$ going forward. We write $\mathcal{H}_h$ for the h^{th} copy of the upper half plane. In this language, $Y^B(\mathcal{O}) = \sqcup_h\Gamma_h\backslash\mathcal{H}_h$. Our objects of study will be an automorphic form $\phi = (\phi_h)_{h\in[H]}\in\bigoplus_h S_k^B(\mathcal{O},\Xi;h)\cong S_k^B(\mathcal{O},\Xi)$, the associated differential forms $\omega = (\omega_{\phi_h})_{h\in[H]}$, where ω_{ϕ_h} is a differential form on $\Gamma_h\backslash\mathcal{H}_h$, and the associated cocycles $\xi = (\xi_{\phi_h})_{h\in[H]}\in\bigoplus_h H^1(\Gamma_h,V_{k,\Xi_h})$. With this notation in hand, we now translate the Hecke action given in Lemma 4.3 into cohomology. Define

$$(T_{\mathfrak{p}}\xi)_h(\gamma) := \sum_j \varpi_{\gamma(j),h}^{-1}\xi_{j(h)}(\varpi_{\gamma(j),h}\gamma\varpi_{j,h}^{-1}). \tag{24}$$

Theorem 5.8. *For $\xi = (\xi_h)_h$ and $\phi = (\phi_h)_h$ as above,*

$$\xi_{(T_{\mathfrak{p}}\phi)_h} = (T_{\mathfrak{p}}\xi)_h\in H^1(\Gamma_h,V_{k,\Xi_h})$$

for all $h\in[H]$.

Proof. For brevity, we write $\varpi_j := \varpi_{j,h}$. Let $(P_h)_{h\in[H]}$ be a collection of (arbitrary) basepoints $P_h\in\mathcal{H}_h$. By Lemma 4.4 that for any $\gamma\in\mathcal{O}_j^\times$, there is a unique $\gamma(j)\in[P]$ for which $\varpi_{\gamma(j)}\gamma\varpi_j^{-1}\in B^\times$, and for this $\gamma(j)$, $\gamma(j)(h) = j(h)$.

In what follows, we think of the action of ϖ_j on $\sqcup\mathcal{H}_h$ as sending $\mathcal{H}_h$ to $\mathcal{H}_{j(h)}$ – this is important so that we only integrate differentials on $\mathcal{H}_h$ against paths on $\mathcal{H}_h$. As discussed below Lemma 4.4, we may assume without loss of generality that $\Xi(\hat{\alpha}_h) = 1$ for all h and that $\varpi_{j,h}$ is locally in $\mathcal{O}_\mathfrak{p} \cap B_\mathfrak{p}^\times$ for every $\mathfrak{p} \nmid \mathfrak{N}$. For $\gamma \in \Gamma_h$,

$$\xi_{(T_\mathfrak{p}\phi)_h}(\gamma) = \sum_j \Xi(\hat{\alpha}_{j(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)\int_{P_h}^{\gamma P_h} \omega_{\phi_{j(h)}|_k\varpi_{j,h}} = \sum_j \varpi_j^{-1}\cdot\int_{\varpi_j P_h}^{\varpi_j\gamma P_h}\omega_{\phi_{j(h)}} = \sum_j \varpi_{\gamma(j)}^{-1}\cdot\int_{\varpi_{\gamma(j)}P_h}^{\varpi_j\gamma P_h}\omega_{\phi_{j(h)}}.$$

We can rewrite this as

$$\sum_j \varpi_{\gamma(j)}^{-1}\cdot\left(\int_{P_{j(h)}}^{\varpi_{\gamma(j)}\gamma\varpi_j^{-1}P_{j(h)}}\omega_{\phi_{j(h)}}\right) + \sum_j \varpi_{\gamma(j)}^{-1}\cdot\left(\int_{\varpi_{\gamma(j)}\gamma\varpi_j^{-1}P_{j(h)}}^{\varpi_{\gamma(j)}\gamma P_h}\omega_{\phi_{j(h)}} + \int_{\varpi_{\gamma(j)}P_h}^{P_{j(h)}}\omega_{\phi_{j(h)}}\right). \tag{25}$$

The first summand in Equation (25) is exactly $(T_\mathfrak{p}\xi)_h$ by Equation (24), so it remains to show that

$$\sum_j \varpi_{\gamma(j)}^{-1}\cdot\left(\int_{\varpi_{\gamma(j)}\gamma\varpi_j^{-1}P_{j(h)}}^{\varpi_{\gamma(j)}\gamma P_h}\omega_{\phi_{j(h)}}\right) + \sum_j \varpi_{\gamma(j)}^{-1}\cdot\left(\int_{\varpi_{\gamma(j)}P_h}^{P_{j(h)}}\omega_{\phi_{j(h)}}\right). \tag{26}$$

is a 1-coboundary. Let $\gamma' := \varpi_{\gamma(j)}\gamma\varpi_j^{-1}$. Substituting this in and applying Equation (21), the first term in Equation (26) becomes

$$\sum_j \varpi_{\gamma(j)}^{-1}\cdot\int_{\gamma' P_{j(h)}}^{\gamma'\varpi_j P_h}\omega_{\phi_{j(h)}} = \sum_j (\varpi_{\gamma(j)}^{-1}\gamma')\cdot\int_{P_{j(h)}}^{\varpi_j P_h}\omega_{\phi_{j(h)}}. \tag{27}$$

Reindexing and noting that $\gamma(j)(h) = j(h)$, the second term in Equation (26) is

$$\sum_j \varpi_{\gamma(j)}^{-1}\cdot\int_{\varpi_j P_h}^{P_{j(h)}}\omega_{\phi_{j(h)}}. \tag{28}$$

The sum of Equation (27) and Equation (28) is then

$$\sum_j (\varpi_{\gamma(j)}^{-1}\gamma' - \varpi_j^{-1})\cdot\int_{P_{j(h)}}^{\varpi_j P_h}\omega_{\phi_{j(h)}} = \sum_j (\gamma - 1)\cdot\left(\varpi_j^{-1}\cdot\int_{P_{j(h)}}^{\varpi_j P_h}\omega_{\phi_{j(h)}}\right).$$

This is a 1-coboundary, so we are done. □

Remark. We can check (e.g. [?]) that Equation (24) is the composition of a restriction, a conjugation, and a corestriction, and as such is a well-defined map on cohomology independent of any choices. It can also be understood geometrically as the map induced on cohomology by a Hecke correspondence.

Let $\mu = (\mu_h)_{h\in[H]}$ be a collection of $\mu_h \in \tilde{\Gamma}_h/\Gamma_h$. As noted earlier, the specific choices of μ_h do not matter. We let μ act on $\xi = (\xi_h)_h$ componentwise.

Lemma 5.9. *The actions of μ and $T_\mathfrak{p}$ on $\bigoplus_h H^1(\Gamma_h, V_{k,\Xi_h})$ commute.*

Proof. By Lemma 4.4, for all $j \in [P]$ there is a permutation $\mu_h\colon [P] \to [P]$ such that $\varpi_{j,h}\mu_h = \mu'_{j(h)}\varpi_{\mu(j),h}$, where $\mu'_{j(h)} \in \tilde{\Gamma}_{j(h)} \setminus \Gamma_{j(h)}$ since $\{\varpi_{j,h}\}_j \subset B_+^\times$. Noting that $\gamma(j)(h) = j(h)$ by Lemma 4.4,

$$\begin{aligned}(T_\mathfrak{p}\cdot(\mu\cdot\xi))_h &= \sum_j \varpi_{\gamma(j)}^{-1}\mu_{j(h)}^{-1}\xi_{j(h)}(\mu_{j(h)}\varpi_{\gamma(j)}\gamma\varpi_j^{-1}\mu_{j(h)}) \\ &= \sum_j (\mu'_h)^{-1}\varpi_{\mu_{j(h)}(\gamma(j))}^{-1}\xi_{j(h)}(\varpi_{\mu_{j(h)}(\gamma(j))}\mu'_h\gamma(\mu'_h)^{-1}\varpi_{\mu_{j(h)}(j)}^{-1}) \\ &= (\mu\cdot(T_\mathfrak{p}\cdot\xi))_h,\end{aligned}$$

where in the last equality we use that the choices of $\{\mu_h\}_h$ do not affect the action of μ on cohomology. □

We can thus compute the action of $T_\mathfrak{p}$ on $S_k^B(\mathcal{O}, \Xi)$ cohomologically, by computing the action on $\bigoplus_h H^1(\Gamma_h, V_{k,\Xi_h})$ using Equation (24) and then projecting to the subspace where μ acts trivially. The actual computation of the cohomology groups is essentially exactly as described in [?, ?, ?]. Let $\Gamma_h(1) := \widehat{\mathcal{O}_0(1)}^1/\mathbb{Z}_F^\times$. By Shapiro's lemma,

$$H^1(\Gamma_h, V_{k,\Xi_h}) \cong H^1\left(\Gamma_1, \operatorname{Ind}_{\Gamma_h}^{\Gamma_h(1)} V_{k,\Xi_h}\right).$$

By [?, Lemma 1.1.4], the Hecke action commutes with Shapiro's lemma for $\mathfrak{p} \nmid \mathfrak{N}$, meaning that by working with the induced representation we can compute Hecke operators for such $\mathfrak{p}$ at level $\mathcal{O}$ while only using the $\{\varpi_{j,h}\}$ of $\mathcal{O}_0(1)$. When $\mathfrak{p} \mid \mathfrak{N}$, we can still compute Hecke operators, but we need to first undo the Shapiro isomorphism, compute the Hecke operator on the level $\mathcal{O}$ space, and then reapply the Shapiro isomorphism. Despite this, we can still get away with using more or less the original $\varpi_{j,h}$, since if $\hat{\alpha}_h \pi_j^{-1} = \varpi_{j,h}^{-1} \hat{\alpha}_{j(h)} \widehat{\mathcal{O}_0(1)}^\times$, then by multiplying $\varpi_{j,h}^{-1}$ on the left by an element of $\mathcal{O}_{j(h)}^\times$, we can find an element of $\mathcal{O}_{j(h)}^\times \varpi_{j,h}$ for which $\hat{\alpha}_h \pi_j^{-1} = \varpi_{j,h}^{-1} \hat{\alpha}_{j(h)} \widehat{\mathcal{O}}^\times$.

6. Definite method

In this section, we describe a procedure for computing $S_k^B(\mathcal{O}, \Xi)$ when B is (totally) definite, i.e. when $r = 0$. Definition 3.1 simplifies in this case, and $M_k^B(\mathcal{O}, \Xi)$ consists of functions $\phi\colon \widehat{B}^\times \to W_k$ such that $\phi(\gamma \hat{x}) = \phi(\hat{x})^\gamma$ and $\phi(\hat{x}\hat{u}) = \Xi(\hat{u})\varphi(\hat{x})$. Algorithms to compute $M_k^B(\mathcal{O}, 1)$ for such B were developed by [?] and extended to fields with arbitrary narrow class number by [?]. In this section, we explain how to extend their methods to arbitrary nebentypus Ξ.

Let $H_1 := \#\operatorname{Cls}\mathcal{O}_0(1)$ and $H := \#\operatorname{Cls}\mathcal{O}$. By Lemma 4.2, $M_k(\mathcal{O}, \Xi) \cong \bigoplus_{h' \in [H]} M_k(\mathcal{O}, \Xi; h')$. Let $\{\hat{\alpha}_h\}_{h \in [H_1]}$ be double coset representatives for $B^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}_0(1)}^\times$. The set $\{\hat{\alpha}_h \hat{\kappa}\}$ – where $h \in [H_1]$ and where $\hat{\kappa}$ ranges over coset representatives for $\widehat{\mathcal{O}_0(1)}^\times / \widehat{\mathcal{O}}^\times$ – can be taken as double coset representatives for $B^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times$. For $h \in [H_1]$, let

$$M_k(\mathcal{O}, \Xi; h)_1 := \bigoplus_{\hat{\kappa} \in \widehat{\mathcal{O}_0(1)}^\times / \widehat{\mathcal{O}}^\times} M_k(\mathcal{O}, \Xi; \hat{\alpha}_h \hat{\kappa}).$$

Because $\mathcal{O} = \mathcal{O}_0(\mathfrak{N})$, there are isomorphisms $\widehat{\mathcal{O}_0(1)}_h^\times / \widehat{\mathcal{O}}_h^\times \cong \widehat{\mathcal{O}_0(1)}^\times / \widehat{\mathcal{O}}^\times \cong \mathbb{P}^1(\mathbb{Z}_F/\mathfrak{N})$. To avoid recomputing the $\{\varpi_{j,h}\}$ at each level, we will work with the induced representation $\operatorname{Ind}_{\widehat{\mathcal{O}}^\times}^{\widehat{\mathcal{O}_0(1)}^\times} \Xi$ (which is isomorphic to $\operatorname{Ind}_{\widehat{\mathcal{O}}_h^\times}^{\widehat{\mathcal{O}_0(1)}_h^\times} \Xi_h$ for every $h \in [H_1]$). This is superficially similar to how we used Shapiro's lemma at the end of Section 6, but note that here the induction is taking place in the domain of the functions in question rather than in the codomain.

Given representatives $\{\hat{\kappa}_x\}_{x \in \mathbb{P}^1(\mathbb{Z}_F/\mathfrak{N})}$ for $\widehat{\mathcal{O}_0(1)}_h^\times / \widehat{\mathcal{O}}_h^\times$, we can take the vectors $[\hat{\kappa}_x]_{x \in \mathbb{P}^1(\mathbb{Z}_F F/\mathfrak{N})}$ to be a basis for $\operatorname{Ind}_{\widehat{\mathcal{O}}^\times}^{\widehat{\mathcal{O}_0(1)}^\times} \Xi$. For any $\hat{\kappa}' \in \widehat{\mathcal{O}_0(1)}_h^\times$ and any coset representative $\hat{\kappa}_x$, there is a unique $\hat{\kappa}'(x) \in \mathbb{P}^1(\mathbb{Z}_F/\mathfrak{N})$ such that $\hat{\kappa}_{\hat{\kappa}'(x)}^{-1} \hat{\kappa}' \hat{\kappa}_x \in \widehat{\mathcal{O}}^\times$. We obtain an explicit realization of $\operatorname{Ind}_{\widehat{\mathcal{O}}^\times}^{\widehat{\mathcal{O}_0(1)}^\times} \Xi$ by defining $\hat{\kappa}'[\hat{\kappa}_x] := \Xi(\hat{\kappa}_{\hat{\kappa}'(x)}^{-1} \hat{\kappa}' \hat{\kappa}_x)[\hat{\kappa}_{\hat{\kappa}'(x)}]$

Lemma 6.1. *For all $h \in [H_1]$, there is a $\mathbb{C}$-linear map*

$$\Phi_h\colon M_k^B(\mathcal{O}, \Xi) \longrightarrow \{F_h\colon \operatorname{Ind}_{\widehat{\mathcal{O}}_h^\times}^{\widehat{\mathcal{O}_0(1)}_h^\times} \Xi_h \longrightarrow W_k | F_h(\gamma \cdot y)^\gamma = F_h(y) \text{ for all } \gamma \in \mathcal{O}_0(1)_h^\times, y \in \widehat{\mathcal{O}_0(1)}_i^\times\}$$

$$\phi \longmapsto (F_h\colon \hat{\alpha}_h \hat{\kappa} \hat{\alpha}_h^{-1} \longmapsto \phi(\hat{\alpha}_h \kappa))_h$$

whose image is isomorphic to $M_k^B(\mathcal{O}, \Xi; h)_1$. In particular,

$$M_k^B(\mathcal{O}, \Xi) \cong \bigoplus_{h \in [H_1]} \operatorname{im} \Phi_h.$$

Proof. We have the following sequence of bijections:

$$B^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times \leftrightarrow \sqcup_h B^\times \backslash B^\times \hat{\alpha}_h \widehat{\mathcal{O}}_0(1)^\times / \widehat{\mathcal{O}}^\times \leftrightarrow \sqcup_h (\hat{\alpha}_h^{-1} B^\times \hat{\alpha}_h \cap \widehat{\mathcal{O}}_0(1)^\times) \backslash \widehat{\mathcal{O}}_0(1)^\times / \widehat{\mathcal{O}}^\times \leftrightarrow \sqcup_h \mathcal{O}_0(1)_h^\times \backslash \widehat{\mathcal{O}_0(1)}_h^\times / \widehat{\mathcal{O}}_h^\times,$$

with corresponding maps (skipping the leftmost)

$$\hat{\alpha}_h \hat{\kappa} \longmapsto \hat{\kappa} \longmapsto \hat{\alpha}_h \hat{\kappa} \hat{\alpha}_h^{-1}.$$

As such, we can think of left $B^\times$-equivariant functions on $\widehat{B}^\times / \widehat{\mathcal{O}}^\times$ as left $\mathcal{O}_h^\times$-equivariant functions on $\widehat{\mathcal{O}_0(1)}_h^\times / \widehat{\mathcal{O}}_h^\times$. When $\mathcal{O} = \mathcal{O}_0(\mathfrak{N})$, there are isomorphisms $\widehat{\mathcal{O}_0(1)}_h^\times / \widehat{\mathcal{O}}_h^\times \cong \widehat{\mathcal{O}_0(1)}^\times / \widehat{\mathcal{O}}^\times \cong \mathbb{P}^1(\mathbb{Z}_F/\mathfrak{N})$. Because we are computing things anyways, we describe $\mathrm{Ind}_{\widehat{\mathcal{O}}_h^\times}^{\widehat{\mathcal{O}_0(1)}_h^\times} \Xi$ by picking representatives $\{\kappa_x\}_{x\in\mathbb{P}^1(\mathbb{Z}_F/\mathfrak{N})}$ for $\widehat{\mathcal{O}_0(1)}_h^\times/\widehat{\mathcal{O}}_h^\times$ and defining basis vectors $[\kappa_x]_{x\in\mathbb{P}^1(\mathbb{Z}_F F/\mathfrak{N})}$. For $\gamma \in \mathcal{O}_0(1)_h^\times$, let $\hat\kappa_\gamma := \hat\alpha_h^{-1}\gamma\hat\alpha_h \in \widehat{\mathcal{O}_0(1)}^\times$. Then,

$$\begin{aligned}(29)\quad F_h(\gamma\cdot(\alpha_h\hat\kappa_x\alpha_h^{-1}))^\gamma &= F_h(\hat\alpha_h\hat\kappa_{\gamma(x)}\hat\alpha_h^{-1}(\hat\alpha_h\hat\kappa_{\gamma(x)}^{-1}\hat\kappa_\gamma\hat\kappa_x)\hat\alpha_h^{-1})^\gamma = \Xi_h(\hat\kappa_{\gamma(x)}^{-1}\hat\kappa_\gamma\hat\kappa_x)\phi(\hat\alpha_h\hat\kappa_\gamma^{-1}\hat\kappa_{\gamma(x)})\\ &= \Xi_h(\hat\kappa_{\gamma(x)}^{-1}\hat\kappa_\gamma\hat\kappa_x)\phi(\hat\alpha_h\hat\kappa_x(\hat\kappa_x^{-1}\hat\kappa_\gamma^{-1}\hat\kappa_{\gamma(x)})) = F_h(\hat\alpha_h\hat\kappa_x\hat\alpha_h^{-1}).\end{aligned}$$

The two spaces have the same dimension and the inverse map sending $(F_i)_i$ to the associated f is an injection, so the spaces are isomorphic. ☐

We now describe how the Hecke operators act through the isomorphism of Lemma 6.1. There is an injection of double coset spaces

$$\widehat{\mathcal{O}}^\times\backslash\widehat{\mathcal{O}}^\times\widehat{\mathcal{O}}^\times \hookrightarrow \widehat{\mathcal{O}_0(1)}^\times\backslash\widehat{\mathcal{O}_0(1)}^\times\hat\pi\widehat{\mathcal{O}_0(1)}^\times$$

because both quotients are trivial away from $\mathfrak{p}$. This injection is a bijection when $\mathfrak{p}\nmid\mathfrak{N}$.

In particular, if we have chosen $\{\hat\pi_j\}_{j\in[P]}$ such that $\widehat{\mathcal{O}}^\times\hat\pi\widehat{\mathcal{O}}^\times = \sqcup_{j\in[P]}\widehat{\mathcal{O}}^\times\hat\pi_j$, then $\widehat{\mathcal{O}_0(1)}^\times\hat\pi\widehat{\mathcal{O}_0(1)}^\times = \sqcup_{j\in[P]}\widehat{\mathcal{O}_0(1)}^\times\hat\pi_j$ if $\mathfrak{p}\nmid\mathfrak{N}$, and otherwise there is one extra coset at level $\mathcal{O}_0(1)$. Removing this extra coset if necessary, we can choose the $\{\hat\pi_j\}$ at level $\mathcal{O}_0(1)$ regardless of which level we ultimately want to compute at.

Before we define the Hecke action, observe that the action of $\widehat{\mathcal{O}_0(1)}^\times$ on $\mathrm{Ind}_{\widehat{\mathcal{O}}^\times}^{\widehat{\mathcal{O}_0(1)}^\times}\Xi$ extends to an action of elements of $\varpi\in\widehat{B}^\times$ for which $\varpi_\mathfrak{N}\in\prod_{\mathfrak{p}|\mathfrak{N}}\mathcal{O}_0(1)_\mathfrak{p}^\times$ – for any $x\in\mathbb{P}_1(\mathbb{Z}_F/\mathfrak{N})$, there exists a $\varpi(x)\in\mathbb{P}_1(\mathbb{Z}_F/\mathfrak{N})$ such that $\hat\kappa_{\varpi(x)}^{-1}\varpi\hat\kappa_x$ is in $\prod_{\mathfrak{p}|\mathfrak{N}}\mathcal{O}_\mathfrak{p}^\times$, and we again define $\varpi\cdot[\hat\kappa_x] := \Xi(\hat\kappa_{\varpi(x)}^{-1}\varpi\hat\kappa_x)[\hat\kappa_{\varpi(x)}]$. With this in hand, we may define

$$(30)\qquad (T_\mathfrak{p}F)_h(\alpha_h\kappa_x\alpha_h^{-1}) := \sum_j F_{j(h)}(\varpi_{j,h}\cdot(\alpha_h\kappa_x\alpha_h^{-1}))^{\varpi_{j,h}}.$$

Theorem 6.2. *With respect to the Hecke actions defined in Equation (5) and Equation (30), the isomorphism of Lemma 6.1 is Hecke equivariant.*

Proof. By Equation (5), for $\phi\in M_k(\mathcal{O},\Xi)$ we have

$$(31)\qquad \Phi(T_\mathfrak{p}\phi)_h(\hat\alpha_h\hat\kappa_x\hat\alpha_h^{-1}) = (T_\mathfrak{p}\phi)(\hat\alpha_h\hat\kappa_x) = \sum_j\phi(\hat\alpha_h\hat\kappa_x\pi_j^{-1}).$$

As in Lemma 4.3, for each j,h, there are elements $\{\varpi_{j,h}\}$ such that

$$(32)\qquad \hat\alpha_h\pi_j^{-1} = \varpi_{j,h}^{-1}\alpha_{j(h)}\widehat{\mathcal{O}_0(1)}^\times.$$

We cannot directly apply Equation (32) to Equation (31) because $\hat\kappa_x$ is in between $\hat\alpha_h$ and $\hat\pi_j^{-1}$. By Equation (3) however, because right multiplication by $\widehat{\mathcal{O}_0(1)}^\times$ permutes the cosets, there exists a permutation x (abusing notation) of $[\mathrm{Nm}(\mathfrak{p})+1]$ such that $\hat\kappa_x\hat\pi_j^{-1} = \hat\pi_{x(j)}^{-1}\hat\kappa'$ for $\hat\kappa'\in\widehat{\mathcal{O}_0(1)}^\times$. Then,

$$(33)\qquad \hat\alpha_h\hat\kappa_x\hat\pi_j^{-1} = \hat\alpha_h\hat\pi_{x(j)}^{-1}\hat\kappa' = \varpi_{x(j),h}^{-1}\hat\alpha_{x(j)(h)}\hat\kappa'' = \varpi_{x(j),h}^{-1}\hat\alpha_{x(j)(h)}\hat\kappa_y\hat u_{j,x,h}$$

for some $y\in\mathbb{P}^1(\mathbb{Z}_F/\mathfrak{N})$, $\kappa,\kappa''\in\widehat{\mathcal{O}_0(1)}^\times$, and $u_{j,x,h}\in\widehat{\mathcal{O}}^\times$. As before, we may assume by weak approximation that the $\{\hat\alpha_h\}_{h\in[H]}$ were chosen so that $(\hat\alpha_h)_\mathfrak{N}\equiv 1 \pmod{\mathfrak{N}}$. Then, restricting Equation (33) to the places at $\mathfrak{N}$

$$\hat\kappa_x \equiv \varpi_{x(j),h}^{-1}\hat\kappa_y\begin{pmatrix} * & * \\ & * \end{pmatrix} \pmod{\mathfrak{N}}.$$

As such, we can write $\varpi_{x(j),h}(x) := y$. Similarly, $(u_{j,x,h})_{\mathfrak{N}} \equiv \hat{\kappa}^{-1}_{\varpi_{x(j)},h}\varpi_{x(j),h}\hat{\kappa}_x \pmod{\mathfrak{N}}$. With this in hand, we can continue Equation (31) as

$$\begin{aligned}\sum_j \phi(\alpha_h \hat{\kappa}_x \hat{\pi}_j^{-1}) &= \sum_j \phi(\varpi^{-1}_{x(j),h} \hat{\alpha}_{x(j)(h)} \hat{\kappa}_{\varpi_{x(j),h}(x)} \hat{u}_{j,x,h}) \\ &= \sum_j \Xi(\hat{\kappa}^{-1}_{\varpi_{x(j),h}(x)} \varpi_{x(j),h} \hat{\kappa}_x) \phi(\hat{\alpha}_{x(j)(h)} \hat{\kappa}_{\varpi_{x(j),h}(x)})^{\varpi_{x(j),h}} \\ &= \sum_j F_{j(h)}(\varpi_{j,h} \cdot \hat{\kappa}_x)^{\varpi_{j,h}}.\end{aligned}$$

□

With Theorem 6.2 in hand, the implementation details of computing Hecke matrices are the same as those that arise in [**?**, **?**, **?**].