COMPUTING HILBERT MODULAR FORMS OF NONPARITIOUS WEIGHT

ABHIJIT S. MUDIGONDA

Abstract. We design and implement an algorithm for computing q-expansion bases of spaces of Hilbert modular forms of nonparitious weight over fields of narrow class number 1. We use this algorithm to compute spaces of Hilbert modular forms over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ of weight $(1, 2)$ with Galois stable levels of norm up to 1500 and quadratic nebentypus. To study this algorithm, we introduce the "elemental" Hecke algebra, a finite algebra generated by rescalings of the usual Hecke operators acting on a space of Hilbert modular forms. The elemental Hecke algebra is equivalent to the usual Hecke algebra in paritious weight, but retains certain rationality properties even in the nonparitious setting even when the usual Hecke algebra is poorly behaved. Using the elemental Hecke algebra, we are also able to present self-contained proofs of some standard facts about Hilbert modular forms that we use in the algorithm.

1. Introduction

If you came to this link looking for examples, please jump to Section 6! The Langlands programme is a sweeping web of conjectures relating automorphic forms, Galois representations, motives, and L-functions. Typically, given an object in one of these four worlds, one hopes to construct corresponding objects in the others. For example, the H^1 of an elliptic curve $E/\mathbb{Q}$ is a motive to which one can associate a compatible family of ℓ-adic Galois representations (the first étale cohomology of E), an L-function, and most nontrivially, an automorphic form. However, not every automorphic form is expected to contribute directly to this story. An automorphic representation over a field K is said to be L-algebraic if its archimedean components satisfy a certain integrality condition. Only L-algebraic automorphic representations are expected ([BG14, Conjecture 3.21]) to have associated compatible systems of ℓ-adic Galois representations valued in ${}^LG(\overline{\mathbb{Q}_\ell})$. It is also conjectured ([BG14, Conjecture 3.15]) that an automorphic representation π is L-algebraic if and only if it is L-arithmetic , i.e. if there is a number field E such that at all unramified primes $\mathfrak{p}$, the Satake parameter of $\pi_\mathfrak{p}$ is defined over E. L-arithmetic representations are particularly conducive to computations as we can perform many computations on them over a fixed number field independent of the primes are interested in.

Much of the existing work on automorphic forms in the context of the Langlands programme focuses on forms whose associated automorphic representations are L-algebraic. The simplest examples of automorphic representations that are not L-algebraic arise from nonalgebraic Hecke characters. One can show that algebraic Hecke characters correspond to compatible families of ℓ-adic Galois characters [Sno09]. In this paper, we will be interested in arguably the second simplest class of nonalgebraic representations, which arise from Hilbert modular forms of nonparitious weight.

1.1. **Hilbert modular forms.** Hilbert modular forms are a natural generalization of classical modular forms (which are automorphic forms for $\mathrm{GL}_2/\mathbb{Q}$) to totally real fields of higher degree. We refer the reader to Section 2.4 for background on Hilbert modular forms. We write $M_k(\mathfrak{N}, \chi)$ for the space of Hilbert modular forms with level $\mathfrak{N} \subset \mathbb{Z}_F$, weight $k \in \mathbb{Z}_{\geq 1}^{[F:\mathbb{Q}]}$ and nebentypus character χ. We say that k is paritious if the entries of k are all congruent modulo 2, and nonparitious otherwise. The theories of Hecke operators and newforms extend to the setting of Hilbert modular forms. The automorphic representation associated to a Hilbert modular newform f is L-algebraic if and only if the weight of f is paritious.

Hilbert modular forms have q-expansions, and because the space $M_k(\mathfrak{N}, \chi)$ is finite-dimensional, we can describe it explicitly by giving the q-expansions (to some precision) of a basis of forms spanning $M_k(\mathfrak{N}, \chi)$. As in the setting of modular forms, $M_k(\mathfrak{N}, \chi)$ is spanned by Hecke eigenforms. As such, we can access the space by first computing matrices for the action of the Hecke operator $T_\mathfrak{p}$ on $M_k(\mathfrak{N}, \chi)$ for sufficiently many $\mathfrak{p}$. By the Jacquet-Langlands correspondence, we can compute these matrices by studying the Hecke action on certain spaces of quaternionic modular forms for a quaternion algebra B/F. Algorithms for producing these matrices when F has narrow class number 1, $k = (2, \ldots, 2)$ (parallel weight two), and $\chi = 1$

(trivial nebentypus) were invented and implemented by Greenberg-Voight [GV11] for indefinite B and by Dembélé [Dem07] for definite B. These methods were extended to the fields of arbitrary narrow class number and general paritious weights by Voight [Voi10] and Dembélé-Donnelly [DD08] respectively. Given matrices for the Hecke action on $M_k(\mathfrak{N}) := M_k(\mathfrak{N}, 1)$ for $k = (2, \ldots, 2)$, Donnelly and Voight [DV21] describe an algorithm for producing a basis of $M_2(\mathfrak{N})$. Their algorithm uses several properties of the finite $\mathbb{Q}$-algebra – the Hecke algebra – generated by the operators $\{T_\mathfrak{p}\}$ acting on $M_k(\mathfrak{N})$. Their method was later generalized to paritious weight k. The main difference in this case is that the base field of the Hecke algebra is a subfield of F^{gal} that is only $\mathbb{Q}$ when the weight is parallel. In forthcoming joint work [ABB$^+$26], we extend all of the above algorithms to deal with an arbitrary nebentypus character χ, but still under the hypothesis of paritious weight. In the definite case, the extension to general nebentypus was described (but to this author's knowledge not implemented) by Dembélé [Dem07].

In this work, we describe and implement an algorithm for computing Hilbert modular forms of nonparitious weight over fields with narrow class number 1.

Theorem 1.1. *Let F be a totally real field of narrow class number 1. Given an integral ideal $\mathfrak{N}$ of F, a weight $k \in \mathbb{Z}_{\geq 1}^{[F:\mathbb{Q}]}$, and a finite order Hecke character χ of F, there is an algorithm that computes a list of q-expansions (to any given precision) of forms spanning $M_k(\mathfrak{N}, \chi)$.*

1.2. **"Elemental" Hecke operators.** The usual Hecke operators $\{T_\mathfrak{p}\}$ are poorly suited to the nonparitious setting, as there is no number field over which they are all defined. It follows that the Hecke eigenvalues of a normalized eigenform are not defined over a fixed number field. The Hecke eigenvalues of a newform determine the Satake parameters of the associated automorphic representation, so this is exactly the failure of L-arithmeticity in this setting. In particular, the methods of [DV21], which use crucially the fact that the Hecke algebra on the new subspace can be written as a product of number fields, do not work here. The main idea in the proof of Theorem 1.1 is the introduction of "elemental" Hecke operators and corresponding "elemental" Hecke algebra. Given a totally positive generator π of $\mathfrak{p}$ (we are assuming that F has narrow class number 1), the elemental Hecke operator T_π is a rescaling of $T_\mathfrak{p}$ that depends on the choice of π but can be defined over a number field independent of $\mathfrak{p}$. We will show that matrices for the action of the elemental Hecke operators on spaces of quaternionic modular forms can be efficiently computed, and that replacing the usual Hecke operators with elemental Hecke operators and the usual Hecke algebra with the elemental Hecke algebra allows us to repair the existing algorithms and compute bases.

In the process, we give alternative proofs of several standard facts (Theorem 5.8, Proposition 5.9, Proposition 5.10) about Hilbert modular forms appearing in e.g. [Shi78]. While these proofs will be of no surprise to the experts, we hope that having self-contained proofs of these results that also work in the nonparitious case will be of some use.

1.3. **Previous computations of nonparitious Hilbert modular forms.** This is not the first work to compute nonparitious Hilbert modular forms. Buzzard [Buz12] computes the Satake parameters of an explicit CM nonparitious Hilbert modular form of weight $(1, 2)$ via automorphic induction. More recently, Dembélé, Loeffler, and Pacetti [DLP19] associate Galois representations to nonparitious Hilbert modular forms and compute some examples of nonparitious forms. To produce examples, they use the definite method of [Dem07] to compute, for $F = \mathbb{Q}(\sqrt{2})$ and $F = \mathbb{Q}(\sqrt{5})$, the action of the Hecke operators $\{T_\mathfrak{p}\}$ on $M_k(\mathfrak{N}, \chi)$. They also use "naive" Hecke operators which are equivalent to our elemental Hecke operators $\{T_\pi\}$, and compute the naive Hecke eigenvalues of a particular nonparitious newform of weight $(4, 3)$.

In our view, the main conceptual difference between this work and the computations of [DLP19] is that we work entirely with the elemental Hecke operators from the beginning, rather than computing the usual Hecke operators and then rescaling. This may seem like an unimportant distinction, but this point of view lets us bypass issues of square roots and work with a number field independent of $\mathfrak{p}$ throughout the computation. Furthermore, this allows us to adapt the methods of [DV21] to produce bases of q-expansions for our spaces.

In practice, when computing spaces in paritious weight at multiple levels, we use various tricks – most of which boil down to some version of Shapiro's lemma – to facilitate efficient computation [DV13]. Our implementation extends these approaches to nonparitious weight in both the definite and indefinite settings, and is integrated with existing machinery for computing tables of Hilbert modular forms. While in some sense these are "implementation details" rather than theoretical differences, these optimizations allow us to efficiently compute spaces of nonparitious forms over higher degree fields and with large levels and weights.

We believe that one of the features of the present work is that it is not an ad hoc implementation, but rather part of the robust package for computing with Hilbert modular forms developed in [ABB+26].

1.4. **Acknowledgments.** I am grateful to my advisor, Frank Calegari, for proposing a relationship between abelian fourfolds of Mumford's type and nonparitious Hilbert modular forms that motivated this project [Cal21]. I learned much of what I know about Hilbert modular forms from him. This work builds on code written by many others, but I'd particularly like to thank Eran Assaf, Edgar Costa, Alex Horawa, Jean Kieffer, and John Voight for their numerous contributions and for humoring my incessant questions. I would also like to thank Mateo Attanasio and Deding Yang for helpful conversations. Last but certainly not least, I am grateful to the ANTS referee who is reading these words for taking the time to look at this paper! I was supported by a Crerar fellowship and an NSF graduate research fellowship during part of this work.

2. Preliminaries

2.1. Symbols.

- $[n]$ – $\{1, \dots, n\}$;
- F – totally real field of narrow class number 1;
- $\mathbb{Z}_F$ – ring of integers of F;
- $\mathbb{Z}_\mathfrak{p}$ – completion of $\mathbb{Z}_F$ at a prime ideal $\mathfrak{p} \subset \mathbb{Z}_F$;
- F^{gal} – Galois closure of F;
- $F_{>0}$ – totally positive elements of F;
- $I_{>0}$ – totally positive elements of an ideal I;
- $\mathbb{Z}_{F,>0}^\times$ – totally positive units of F;
- $\mathfrak{d}_F^{-1}$ – codifferent of F;
- $\mathrm{GL}_2^+(F)$ – elements of $\mathrm{GL}_2(F)$ with totally positive determinant;
- $\mathcal{H}$ – complex upper half-plane;
- B – quaternion algebra with center F;
- $B_\infty^\times$ – $\prod'_{v\mid\infty} B_v^\times$;
- $\widehat{B}^\times$ – $\prod_{v\nmid\infty} B_v^\times$;
- $\mathcal{O}$ – (Eichler) order in B;
- $\widehat{\mathcal{O}}^\times$ – $\prod_{v\nmid\infty} \mathcal{O}_v^\times$.

2.2. **Embeddings and multi-index notation.** We fix once and for all an embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. In particular, this restricts to an embedding $\iota\colon F^{\mathrm{gal}} \hookrightarrow \mathbb{R}$. We also fix an ordering $(\sigma_i)_{i\in[n]}$ of the n embeddings of F into F^{gal}. Given an element $x \in F$, we write $x_i := \sigma_i(x) \in F^{\mathrm{gal}}$. Similarly, for a matrix $\gamma \in M_2(F)$, we write $\gamma_i \in M_2(F^{\mathrm{gal}})$ for the matrix obtained by applying σ_i entrywise.

Given a totally positive element $x \in F^{\mathrm{gal}}$ we define $x^{1/s}$ to be the unique s^{th} root y of x in $\overline{\mathbb{Q}}$ such that $\iota(y) \in \mathbb{R}_{>0}$. We can extend this to define $x^{r/s}$ for any totally positive $x \in F^{\mathrm{gal}}$ and $\frac{r}{s} \in \mathbb{Q}$. We will make frequent use of multi-index notation. For $t \in \mathbb{Q}^n$ and $x \in F^{\mathrm{gal}}$, we write $x^t := \prod_i x_i^{t_i} = \prod_i \sigma_i(x)^{t_i} \in \overline{\mathbb{Q}}$. Similarly, for $z \in \mathbb{C}^n$ and $t \in \mathbb{Z}^n$, we write $z^k := \prod_i z_i^{t_i}$. For $t \in \mathbb{Q}$ and $x \in F^{\mathrm{gal}}$, we write $x^{\underline{t}} := \prod_i x_i^t$.

2.3. **Hecke characters.** We refer to [Shu] for relevant background. A Hecke character on a field K is a character $\chi\colon K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$. Because $\mathbb{C}^\times$ has no small subgroups, there exists an open subgroup $U \subset \hat{K}^\times := \prod'_\mathfrak{p} K_\mathfrak{p}$ on which $\chi|_{\hat{K}^\times}$ is trivial. The intersection $U \cap K$ is an ideal $\mathfrak{N} \subset \mathbb{Z}_K$ which we call the conductor of χ and denote by $\mathrm{cond}(\chi)$.

Choose a uniformizer at every finite place – this choice will not matter for our application, and does not matter at all at primes coprime to the conductor. Writing $\mathbb{Z}_{K,\mathfrak{p}}$ for the completion of $\mathbb{Z}_K$ at $\mathfrak{p}$, we decompose the idèle group as

$$\mathbb{A}_K^\times = \left((\mathbb{R}^\times)^r \times (\mathbb{C}^\times)^s\right) \times I_K \times \prod_{\mathfrak{p}\subset\mathbb{Z}_K} \mathbb{Z}_{K,\mathfrak{p}}^\times, \tag{1}$$

where I_K is the group of fractional ideals of K. As such, we can decompose any Hecke character χ as a product $\chi_\infty \cdot \chi_I \cdot \chi_0$, where the three factors are the restriction of χ to the three factors in Equation (1), respectively. In particular, χ_0 is a Dirichlet character of K whose conductor is $\mathrm{cond}(\chi)$. We can also extend χ_0 to a character of $\hat{K}^\times$ by defining it to be 1 on I_K. We also write $\chi_I^*\colon I_F \to \mathbb{C}$ for the function which is $\chi_I(J)$ on any ideal $J \subset \mathbb{Z}_K$ coprime to $\mathrm{cond}(\chi)$ and is 0 otherwise.

We say that a Hecke character is finite order if χ_∞ is finite order. In this case, $\chi_\infty\chi_0$ determines χ_I on principal ideals and there will be $\#\mathrm{Cl}_K$ extensions of $\chi_\infty\chi_0$ to χ – there always exists such an extension, and any two extensions χ and χ' differ by a character of Cl_K.

2.4. **Hilbert modular forms.** Let F be a totally real number field of degree $n > 1$ of narrow class number 1. For $z \in \mathcal{H}^n$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F)$, we define

$$\gamma z := \left(\frac{\iota(a_i)z_i + \iota(b_i)}{\iota(c_i)z_i + \iota(d_i)}\right)_{i\in[n]} \in \mathcal{H}^n \qquad \text{and} \qquad j(\gamma, z) := (\iota(c_i)z_i + \iota(d_i))_{i\in[n]} \in \mathbb{C}^n.$$

Given a function $f\colon \mathcal{H}^n \to \mathbb{C}$, $k \in \mathbb{Z}^n_{>0}$, and $\gamma \in \mathrm{GL}_2^+(F)$, we can define another function

$$(f|_k\gamma)(z) := \frac{(\det\gamma)^{k/2}}{j(\gamma,z)^k} f(\gamma z).$$

Let $\mathfrak{N} \subset \mathbb{Z}_F$ be an ideal, and set

$$\mathcal{O}_0(\mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_F) \colon c \in \mathfrak{N} \right\} \qquad \text{and} \qquad \Gamma_0(\mathfrak{N}) := \mathcal{O}_0(\mathfrak{N}) \cap \mathrm{GL}_2^+(F).$$

For a finite order character χ of modulus $\mathfrak{N}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O}_0(\mathfrak{N})$, we define $\chi_0(\gamma) := \chi_0(d)$.

Definition 2.1. Let $\mathfrak{N}$ and χ be as above, and fix $k \in \mathbb{Z}^n_{\geq 1}$. A Hilbert modular form of weight k, level $\mathfrak{N}$, and nebentypus χ is a holomorphic function $f\colon \mathcal{H}^n \to \mathbb{C}$ such that for any $\gamma \in \Gamma_0(\mathfrak{N})$, $f|_k\gamma(z) = \chi_0(\gamma)f(z)$. We write $M_k(\mathfrak{N}, \chi)$ for the complex vector space of Hilbert modular forms of level $\mathfrak{N}$ and nebentypus χ.

Remark. It may seem strange that the condition on f depends only on χ_0 and not on all of χ. However, as noted in Section 2.3, for F with (narrow) class number 1, χ_0 determines χ and there is no distinction. For a discussion of the general case, see [ABB$^+$26].

A weight k is parallel if the entries of k are all the same and paritious if the entries of k are all congruent modulo 2. Given a weight k, we write k_i for the i^{th} component of k and $k_0 := \max_i k_i$. The space $M_k(\mathfrak{N}, \chi)$ is a direct sum of the subspace of cusp forms $S_k(\mathfrak{N}, \chi)$ and the subspace of Eisenstein series, as defined in [Shi78]. There are explicit formulas for the Eisenstein series in $M_k(\mathfrak{N}, \chi)$ (see [DK21]), so the problem of computing a basis for $M_k(\mathfrak{N}, \chi)$ essentially reduces to computing a basis for $S_k(\mathfrak{N}, \chi)$. When k is nonparallel there are no Eisenstein series at all, and $M_k(\mathfrak{N}, \chi) = S_k(\mathfrak{N}, \chi)$.

Any $f \in M_k(\mathfrak{N}, \chi)$ has a Fourier expansion

$$f(z) := \sum_{\nu \in \mathfrak{d}^{-1}_{F,>0}} a_\nu(f) \exp\left(2\pi i \sum_j \iota(\nu_j) z_j\right).$$

We call $a_\nu := a_\nu(f)$ the Fourier coefficient of f at ν. Applying the condition in Definition 2.1 to the matrices $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$ for $\epsilon \in \mathbb{Z}^\times_{F,>0}$, we find

$$a_{\epsilon\nu} = \epsilon^{k/2} a_\nu \text{ for all } \epsilon \in \mathbb{Z}^\times_{F,>0} \tag{2}$$

Similarly, applying it to $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ for $\epsilon \in \mathbb{Z}^\times_F$,

$$\chi(\epsilon) = \mathrm{sign}(\epsilon)^k := \prod_i \mathrm{sign}(\epsilon_i)^{k_i} \text{ for all } \epsilon \in \mathbb{Z}^\times_F. \tag{3}$$

To any $\nu \in \mathfrak{d}^{-1}_{F,>0}$, we associate an integral ideal $\mathfrak{n} := (\nu)\mathfrak{d}_F$. The ideal $\mathfrak{n}$ then has a corresponding "ideal coefficient"

$$a_\mathfrak{n}(f) := a_\nu(f)\nu^{(k_0 - k)/2}. \tag{4}$$

This is well-defined by Equation (2), as replacing ν by $\epsilon\nu$ for $\epsilon \in \mathbb{Z}^\times_{F,>0}$ does not change the value of $a_\mathfrak{n}(f)$.

Given an ideal $\mathfrak{m} \subset \mathbb{Z}_F$, there is an explicit formula for the Hecke operator $T_\mathfrak{m}$ on $f \in M_k(\mathfrak{N}, \chi)$ in terms of the ideal coefficients [Shi78]:

$$a_\mathfrak{n}(T_\mathfrak{m} f) = \sum_{\mathfrak{m}+\mathfrak{n} \subset \mathfrak{a}} \chi^*(\mathfrak{a}) \,\mathrm{Nm}(mfa)^{k_0 - 1} a_{\mathfrak{n}\mathfrak{m}\mathfrak{a}^{-2}}(f). \tag{5}$$

The Hecke operators satisfy the following relations.

(6) $$T_{\mathfrak{p}^t} = T_{\mathfrak{p}} T_{\mathfrak{p}^{t-1}} - \mathrm{Nm}(\mathfrak{p})^{k_0-1}\chi^*(\mathfrak{p}) T_{\mathfrak{p}^{t-2}} \qquad \text{and} \qquad T_{\mathfrak{n}\mathfrak{m}} = T_{\mathfrak{n}} T_{\mathfrak{m}} \text{ if } (\mathfrak{n},\mathfrak{m}) = 1.$$

Let $F^{\mathrm{gal}}(\chi)$ denote the compositum of F^{gal} and the cyclotomic field in which χ_0 is valued. We can think of the components of k as being indexed by the real embeddings of F. Then, $\mathrm{Gal}(F^{\mathrm{gal}}/\mathbb{Q})$ acts on k by permuting the components, and we write $F' \subset F^{\mathrm{gal}}$ for the field fixed by automorphisms of F^{gal} that preserve k ([Shi78, Proposition 1.4]).

We say that a set of linear operators $\{T\}$ acting on a finite-dimensional complex vector space V can be defined over a field K if there is a choice of basis of V such that in this basis, the matrix of every operator in $\{T\}$ has entries in K.

Theorem 2.2 (Implicit in [Shi78])**.** *The Hecke operators $\{T_{\mathfrak{p}}\}$ acting on $M_k(\mathfrak{N},\chi)$ can be defined over $F'(\chi)$ when k is paritious.*

Remark. In many practical settings – for example when F is a quadratic or Galois cubic field and χ is trival or quadratic – we have $F'(\chi) = F$. One can also check that $F'(\chi)$ is the smallest possible coefficient field of a Hilbert modular form in $S_k(\mathfrak{N},\chi)$ can be defined, as the coefficient field will always contain the nebentypus field and by Equation (2) must contain $\epsilon^{k/2}$ for any $\epsilon \in \mathbb{Z}_F^\times$.

For any $\mathfrak{M} \subset \mathbb{Z}_F$ and $\mathfrak{D} \subset \mathbb{Z}_F$, there is a degeneracy map

(7) $$\iota_{\mathfrak{D}} \colon M_k(\mathfrak{M},\chi) \hookrightarrow M_k(\mathfrak{M}\mathfrak{D},\chi)$$

given by

(8) $$a_{\mathfrak{n}}(\iota_{\mathfrak{D}}(f)) := \begin{cases} a_{\mathfrak{n}\mathfrak{D}^{-1}}(f) & \mathfrak{D}|\mathfrak{n} \\ 0 & \text{otherwise.} \end{cases}$$

We write $M_k(\mathfrak{N},\chi)^{\mathrm{new}}$ to denote the complement in $M_k(\mathfrak{N},\chi)$ of the sum of the images of the degeneracy maps $\iota_{\mathfrak{D}}$ over all $\mathfrak{D}|\mathfrak{N}$ such that $\mathrm{cond}(\chi)|\mathfrak{N}\mathfrak{D}^{-1}$. Forms in $M_k(\mathfrak{N},\chi)^{\mathrm{new}}$ are called newforms. The following is a Hilbert modular forms analogue of Atkin-Lehner-Li theory for modular forms, and is proved in the same way.

Theorem 2.3. *There is a decomposition*

$$M_k(\mathfrak{N},\chi) \cong \bigoplus_{\substack{\mathfrak{M}|\mathfrak{N} \\ \mathrm{cond}(\chi)|\mathfrak{M}}} \bigoplus_{\mathfrak{D}|\mathfrak{N}\mathfrak{M}^{-1}} \iota_{\mathfrak{D}}(M_k(\mathfrak{M},\chi)^{\mathrm{new}}).$$

When k is paritious, we define the Hecke algebra $\mathbb{T} := \mathbb{T}_{F'(\chi)}(\mathfrak{N},k,\chi)$ to be the commutative $F'(\chi)$-algebra generated by the Hecke operators $\{T_{\mathfrak{m}}\}_{\mathfrak{m}\subset\mathbb{Z}_F}$ acting on $M_k(\mathfrak{N},\chi)$. The "anemic" Hecke algebra $\mathbb{T}_0 := (\mathbb{T}_{F'(\chi)})_0$ is defined similarly but is generated by the Hecke operators $\{T_{\mathfrak{m}} \colon (\mathfrak{m},\mathfrak{N}) = 1\}$. By the same argument as is used to prove Theorems 5.5.3 and 5.8.2 of [DS05], one can check that $\mathbb{T}_0$ acts semisimply on $M_k(\mathfrak{N},\chi)$ and that $\mathbb{T}$ acts semisimply on $M_k(\mathfrak{N},\chi)^{\mathrm{new}}$. Therefore, $M_k(\mathfrak{N},\chi)^{\mathrm{new}}$ has a basis of $\mathbb{T}$-eigenforms. We say that a $\mathbb{T}$-eigenform f is normalized if the ideal coefficient $a_{(1)}(f)$ is 1. By Equation (5), the coefficient $a_{\mathfrak{p}}(f)$ of a normalized eigenform f is exactly the eigenvalue of $T_{\mathfrak{p}}$ on f.

Proposition 2.4 ([Shi78, Proposition 2.8])**.** *Let $f \in S_k(\mathfrak{N},\chi)$ be a normalized eigenform with k paritious. Then, $\mathbb{Q}(\{a_{\mathfrak{p}}(f)\})$ is a finite extension.*

Proposition 2.5 (Special case of [Shi78, Proposition 2.6])**.** *If $f \in S_k(\mathfrak{N},\chi)$ and $\tau \in \mathrm{Aut}(\mathbb{C})$ fixes $F'(\chi)$, then*

$$^{\tau}f(z) := \sum_{\nu \in \mathfrak{d}_{F,>0}^{-1}} \tau(a_\nu(f)) \exp\left(2\pi i \sum_j \iota(\nu_j) z_j\right)$$

is an element of $S_k(\mathfrak{N},\chi)$. Furthermore, if f is a $\mathbb{T}_0$ or $\mathbb{T}$-eigenform, then so is $^{\tau}f$.

In Section 5, we give different proofs of Theorem 2.2, Proposition 2.4, and Proposition 2.5.

3. Computing spaces of Hilbert modular forms of paritious weight

Our goal in this section is to compute a basis of $S_k(\mathfrak{N},\chi)$ (i.e. produce explicit q-expansions of forms spanning the space). Following the approach of [DV21], we proceed as follows:

(1) Use Theorem 3.1 to compute "full" Hecke matrices for the $T_\mathfrak{p}$ action on $S_k(\mathfrak{M},\chi)$ for $\mathfrak{M}\subset\mathbb{Z}_F$ such that $\mathrm{cond}(\chi)|\mathfrak{M}$ and $\mathfrak{M}|\mathfrak{N}$.
(2) Use these matrices and rationality properties of the Hecke algebra $\mathbb{T}$ to cut out a subspace V_f for each Galois conjugacy classes of newforms.
(3) Use Equation (6) to compute matrices for $\{T_\mathfrak{m}\}$, where $\mathfrak{m}\subset\mathbb{Z}_F$ ranges over all ideals, from the matrices $\{T_\mathfrak{p}\}$.
(4) Use properties of $\mathbb{T}$ to compute q-expansions with coefficients in $F'(\chi)$ spanning each V_f.
(5) Use Theorem 2.3 and Equation (8) to assemble a basis for $S_k(\mathfrak{N},\chi)$ over $F'(\chi)$.

A key point is that throughout, all our computations are over the field $F'(\chi)$. In particular, the field we work over is independent of which Hecke operators $T_\mathfrak{p}$ we are working with. In practice, we want to compute the q-expansions of a basis up to some precision, and as such only compute finitely many Hecke operators $\{T_\mathfrak{p}\}$. We suppress such considerations in what follows.

3.1. An algorithm for computing $S_k(\mathfrak{N},\chi)$ for k paritious.

3.1.1. *Compute "full" Hecke matrices.*

Theorem 3.1. *There is an algorithm, which given a weight $k\in\mathbb{Z}^n_{\geq 2}$, a level $\mathfrak{N}\subset\mathbb{Z}_F$, a finite order nebentypus χ of modulus $\mathfrak{N}$, and a prime $\mathfrak{p}$, returns a matrix for the Hecke operator $T_\mathfrak{p}$ on $S_k(\mathfrak{N},\chi)$ over $F'(\chi)$ in a basis independent of $\mathfrak{p}$.*

In Section 4, we will prove Theorem 3.1 by computing the Hecke action on a more tractable space that is isomorphic as a Hecke module to $S_k(\mathfrak{N},\chi)$. Applying Theorem 3.1 to every $\mathfrak{M}|\mathfrak{N}$ such that $\mathrm{cond}(\chi)|\mathfrak{M}$, we can compute matrices for the action of $T_\mathfrak{p}$ on $S_k(\mathfrak{M},\chi)$ for any $\mathfrak{M}$ and any $\mathfrak{p}$.

3.1.2. *Restrict the Hecke matrices to Galois orbits of newforms.* We first want to use the full Hecke matrices to identify the subspace $S_k(\mathfrak{N},\chi)^{\mathrm{new}}\subset S_k(\mathfrak{N},\chi)$. Let $\mathfrak{M}|\mathfrak{N}$ be such that $\mathrm{cond}(\chi)|\mathfrak{M}$ and choose $\mathfrak{D}|\mathfrak{N}\mathfrak{M}^{-1}$. One can check from Equation (5) and Equation (8) that the subspaces $\iota_\mathfrak{D}(S_k(\mathfrak{M},\chi)^{\mathrm{new}}\subset S_k(\mathfrak{M},\chi)$ in Theorem 2.3 are closed under the action of $\mathbb{T}_0 := (\mathbb{T}_{F'(\chi)})_0$ and furthermore that $S_k(\mathfrak{M},\chi)^{\mathrm{new}}$ is isomorphic as a $\mathbb{T}_0$-module to $\iota_\mathfrak{D}\left(S_k(\mathfrak{M},\chi)^{\mathrm{new}}\right)$. For a level $\mathfrak{M}\subset\mathbb{Z}_F$ and a Hecke operator $T\in\mathbb{T}$, let $\mu_{\mathfrak{M},T_\mathfrak{p}}$ be the squarefree part of the characteristic polynomial of $T_\mathfrak{p}$ acting on $S_k(\mathfrak{M},\chi)$. Because $\mathbb{T}_0$ acts semisimply on $S_k(\mathfrak{N},\chi)$, we deduce the following.

Proposition 3.2 ([DV21])**.**
$$S_k(\mathfrak{N},\chi)^{\mathrm{new}} = \bigcap_{\substack{\mathfrak{p}\subset\mathbb{Z}_F \text{ prime}\\ \mathfrak{p}\nmid\mathfrak{N}}}\bigcap_{\mathfrak{M}}\mathrm{im}\,\mu_{\mathfrak{M},T_\mathfrak{p}},$$
where the intersection is over $\mathfrak{M}$ such that $\mathfrak{M}|\mathfrak{N}$ and $\mathrm{cond}(\chi)|\mathfrak{M}$.

Using Proposition 3.2, we can restrict our full Hecke matrices to the new subspace – if we compute at the smaller levels first, we can compute the dimension of $S_k(\mathfrak{N},\chi)^{\mathrm{new}}$ using Theorem 2.3, and can compute the intersection of Proposition 3.2 at various primes until we reach the correct dimension.

Because $\mathbb{T} := \mathbb{T}_{F'(\chi)}$ acts on $S_k(\mathfrak{N},\chi)^{\mathrm{new}}$ as a finite commutative $F'(\chi)$-algebra, $\mathbb{T}_{F'(\chi)}\cong\prod_f K_f$ for field extensions $K_f/F'(\chi)$. We then have the decomposition

$$S_k(\mathfrak{N},\chi)^{\mathrm{new}}\cong\bigoplus_f V_f. \tag{9}$$

One can check that the elements of $\mathrm{Hom}_{F'(\chi)}(\mathbb{T},\mathbb{C})$ are in bijection with normalized Hecke eigenforms. Because any such algebra homomorphism is an element of $\mathrm{Hom}_{F'(\chi)}(K_f,\mathbb{C})$ for some f, we deduce that the sum in Equation (9) is indexed by representatives of Galois orbits of newforms under $\tau\in\mathrm{Aut}_{F'(\chi)}(\mathbb{C})$. Proposition 2.4 and Proposition 2.5 follow from this.

Because $\mathbb{T}$ acts semisimply on $M_k(\mathfrak{N},\chi)^{\mathrm{new}}$ and is generated by a single element $T\in\mathbb{T}$ (not necessarily a $T_\mathfrak{p}$, but some element nonetheless), we have the decomposition

$$M_k(\mathfrak{N},\chi)^{\mathrm{new}}\cong\oplus_{\mu'|\mu(\mathfrak{N},T)}\ker\mu'$$

where the sum ranges over the factors μ' of $\mu(\mathfrak{N}, T)$ over $F'(\chi)$. Since T is a generator for $\mathbb{T}$, the factors μ' are in bijection with the factors K_f of $\mathbb{T}$ and the subspaces $\ker \mu'$ are in bijection with the subspaces V_f.

3.1.3. *Produce matrices of $\{T_\mathfrak{m}\}$ from the matrices of $\{T_\mathfrak{p}\}$.* For each V_f, we can apply the identities in Equation (6) to compute $T_\mathfrak{m}|_{V_f}$ for any ideal $\mathfrak{m} \subset \mathbb{Z}_F$. In practice, we want to compute $\{T_\mathfrak{m}|_{V_f} : \mathfrak{m} \subset \mathbb{Z}_F, \mathrm{Nm}(\mathfrak{m}) \leq X\}$ for some bound X. Using the identities in Equation (6), and processing the ideals $\mathfrak{m}$ in order of the number of prime factors of $\mathfrak{m}$ (with multiplicity), we can use dynamic programming to compute $\{T_\mathfrak{m}|_{V_f}\}$ with one additional matrix multiplication per ideal $\mathfrak{m}$.

3.1.4. *Compute q-expansions of an $F'(\chi)$-basis of each newform orbit.* Each V_f is a simple $\mathbb{T}$-module on which $\mathbb{T}$ acts as a field extension $K_f/F'(\chi)$. This K_f is exactly the coefficient field of the newform orbit representative f). Let T be a generator of $\mathbb{T}|_{V_f}$. Letting $d := [K_f : F'(\chi)]$, given any $g \in V_f$, $\{T^j g\}_{j=0}^{d-1}$ is a basis for V_f. We will take $g := \sum_{\tau \in \mathrm{Hom}_{F'(\chi)}(K_f, \mathbb{C})} {}^\tau f$. Then, g has coefficients in $F'(\chi)$. The following lemma lets us compute the ideal coefficients of $\{T^j g\}_{j=0}^{d-1}$.

Lemma 3.3. $a_\mathfrak{n}(T^j g) = \mathrm{tr}(T^j T_\mathfrak{n})$.

Proof.

$$\mathrm{tr}(T^j T_\mathfrak{n}) = \sum_\tau \lambda_T(f_\tau)^j a_\mathfrak{n}({}^\tau f) \; = a_\mathfrak{n}\left(\sum_\tau \lambda_T(f_\tau)^j ({}^\tau f)\right) \; = a_\mathfrak{n}(T^j g).$$

□

Because the coefficients of g are in $F'(\chi)$, we can use Lemma 3.3 to produce q-expansions $\{T^j g\}_{j=0}^{d-1}$ spanning V_f with coefficients in $F'(\chi)$. In particular, we can do this directly from the matrix $T \in \mathbb{T}$, which has entries in $F'(\chi)$. As such, all of our computation can be performed over the field $F'(\chi)$ – we never actually work with the coefficient fields $K_f/F'(\chi)$.

3.1.5. *Assemble the bases of newform orbits to produce a basis for $S_k(\mathfrak{N}, \chi)$.* Repeat the previous three steps to produce the q-expansions of a basis of $S_k(\mathfrak{M}, \chi)$ for all $\mathfrak{M}|\mathfrak{N}$ such that $\mathrm{cond}(\chi)|\mathfrak{M}$. Applying Theorem 2.3 with the degeneracy maps as in Equation (7), we obtain the q-expansions of a basis of $S_k(\mathfrak{N}, \chi)$ with coefficients in $F'(\chi)$.

3.2. **Forms of partial weight one.** Theorem 3.1 does not let us access Hecke matrices on spaces of Hilbert modular forms of partial weight one. For these, we can apply the Hecke stability method of Schaeffer [Sch15] (see also [MS15, ABB+26] for the extension to Hilbert modular forms). Concretely, choosing an Eisenstein series $E \in M_l(\mathfrak{N}, \psi)$ nonvanishing at the cusp at infinity, $S_k(\mathfrak{N}, \chi)$ is contained in the Hecke stable subspace U of the space of meromorphic modular quotients $V := \frac{S_{k+l}(\mathfrak{N}, \chi\psi)}{E}$. We can efficiently compute V using the fast multiplication and division algorithms in [ABB+26], and can compute U from V using the formula in Equation (5). In the case of classical modular forms, Schaeffer proves that $S_k(\mathfrak{N}, \chi) = U$, i.e. that forms in U are in fact holomorphic. We expect his proof to generalize to the Hilbert modular setting, but do not assume this. Instead, we can compute U, and verify that forms $f \in U$ are holomorphic by checking that their squares (which have weight in $\mathbb{Z}_{\geq 2}^n$) lie in $S_{2k}(\mathfrak{N}, \chi^2)$ and are hence holomorphic.

4. Quaternionic modular forms and computing matrices for $T_\mathfrak{p}$

In this section, we will prove Theorem 3.1 by establishing a Hecke module isomorphism between $M_k(\mathfrak{N}, \chi)$ and an appropriate space of quaternionic modular forms on which Hecke matrices can be computed explicitly. In this section, we follow the framework (and where possible, the notation) of Section 7 of [DV13]. Our presentation is also substantially influenced by [Mar20] and [Dem07].

Let B/F be a quaternion algebra with discriminant $\mathrm{disc}(B)$. Let r (resp. s) be the number of infinite places split (resp. ramified) in B. Fix a level $\mathfrak{M} \subset \mathbb{Z}_F$ such that $(\mathfrak{M}, \mathrm{disc}(B)) = (1)$ and a finite order Hecke character χ with $\mathrm{cond}(\chi)|\mathfrak{M}$. Let $\mathbb{Z}_\mathfrak{M} := \prod_{\mathfrak{p}|\mathfrak{M}} \mathbb{Z}_\mathfrak{p}$. Given a splitting $\iota_\mathfrak{M} : B \hookrightarrow M_2(\prod_{v|\mathfrak{M}} F_v)$, we define the Eichler order of level $\mathfrak{M}$,

$$\mathcal{O}_0(\mathfrak{M}) := \{x \in B : \; \iota_\mathfrak{M}(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_\mathfrak{M}), \; c \in \mathfrak{M}\mathbb{Z}_\mathfrak{M}\}.$$

To avoid clutter, we write $\mathcal{O} := \mathcal{O}_0(\mathfrak{M})$ when there is no ambiguity. For $\hat{x} \in \widehat{B}^\times$ such that

$$(\hat{x})_{v|\mathfrak{M}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_{\mathfrak{M}}) \tag{10}$$

we define $\chi_0(\hat{x}) := \chi_0(d)$. Similarly, if the image of $x \in \widehat{B}^\times$ under the diagonal embedding to $\widehat{B}^\times$ is a $\hat{x}$ satisfying Equation (10), we define $\chi_0(x) := \chi_0(\hat{x})$. Choose a splitting

$$B^\times \hookrightarrow \prod_{\substack{v|\infty \\ v \text{ ramified}}} B_v^\times \hookrightarrow \mathrm{GL}_2(\mathbb{C})^s.$$

Precomposing with this splitting, the right $\mathrm{GL}_2(\mathbb{C})^s$-representation

$$W_k(\mathbb{C}) := \bigotimes_{\substack{v|\infty \\ v \text{ ramified}}} \left(\mathrm{Sym}^{k_v-2}\,\mathbb{C}^2 \otimes (\det)^{(\underline{k_0}-k_v)/2}\right) \tag{11}$$

gives rise to a representation of $B^\times$ over $\mathbb{C}$. In practice, we actually pick a number field K containing F^{gal} and splitting B, and do all our computation over the compositum $K(\chi)$ containing K and the field of values of χ.

Definition 4.1. A quaternionic modular form over $B^\times$ of level $\mathfrak{M}$ and nebentypus χ is a function

$$\phi\colon \mathcal{H}^r \times \widehat{B}^\times \longrightarrow W_k(\mathbb{C})$$

such that

(1) For any $\gamma \in B_+^\times$,

$$\phi(z, \hat{x}) = \prod_{i=1}^{r} \frac{\det \gamma_i^{k_i/2}}{j(\gamma_i, z_i)^{k_i}} \phi(\gamma z, \gamma \hat{x})^\gamma.$$

(2) For any $\hat{u} \in \widehat{\mathcal{O}}^\times$,

$$\phi(z, \hat{x}\hat{u}) = \chi_0(\hat{u})\phi(z, \hat{x}).$$

(3) ϕ is holomorphic in the first variable.

The space of quaternionic modular forms over $B^\times$ of level $\mathfrak{M}$ and nebentypus χ is denoted $M_k^B(\mathfrak{M}, \chi)$.

As in Section 2.4, $M_k^B(\mathfrak{M}, \chi)$ contains a subspace of quaternionic cusp forms $S_k^B(\mathfrak{M}, \chi)$. When B/F is a quaternion division algebra (i.e. not $M_2(F)$), $S_k^B(\mathfrak{M}, \chi) = M_k^B(\mathfrak{M}, \chi)$ unless $r = 0$ and k is parallel. Even if $S_k^B(\mathfrak{M}, \chi) \subsetneq M_k^B(\mathfrak{M}, \chi)$, it is easy to compute and understand the complement of $S_k^B(\mathfrak{M}, \chi)$. As such, computing Hecke matrices on $M_k^B(\mathfrak{M}, \chi)$ is essentially equivalent to computing Hecke matrices on $S_k^B(\mathfrak{M}, \chi)$.

To make use of the conditions in Definition 4.1, it will be helpful to understand the double coset space $B_+^\times \backslash \widehat{B}^\times / \mathcal{O}^\times$.

Theorem 4.2 ([Voi21, 28.4.3, 27.7.1, 28.5.5])**.**

(1) The map

$$\begin{aligned} F\colon\ B_+^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times &\longrightarrow \mathrm{Cls}\,\mathcal{O} \\ B_+^\times \hat{\alpha} \hat{\mathcal{O}}^\times &\longmapsto \hat{\alpha}\hat{\mathcal{O}} \cap B =: I_\alpha \end{aligned}$$

is a bijection.

(2) The map

$$\mathrm{nrd}\colon B_+^\times \backslash \widehat{B}^\times / \mathcal{O}^\times \longrightarrow F_+^\times \backslash \widehat{F}^\times / \mathrm{nrd}(\hat{\mathbb{Z}}_F^\times) \cong \mathrm{Cl}_F^+ \tag{12}$$

is a surjection, and if B is indefinite, then it is a bijection.

Let H be the the cardinality of this double coset space, and pick representatives $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_H\} \subset \widehat{B}^\times$. Given $\phi \in M_k^B(\mathfrak{M}, \chi)$ and $h \in [H]$, we can define a function

$$\begin{aligned} \phi_h\colon\ \mathcal{H}^s &\longrightarrow W_k(\mathbb{C}) \\ z &\longmapsto f(z, \hat{\alpha}_h) \end{aligned}.$$

For any function $\phi_h \colon \mathcal{H}^r \to W_k(\mathbb{C})$, we define an action

$$\phi_h|_k\gamma := \frac{(\det\gamma)^{k/2}}{j(\gamma,z)^k}\phi_h(\gamma z)^\gamma. \tag{13}$$

Let $\mathcal{O}_h := \hat{\alpha}_h\hat{\mathcal{O}}\hat{\alpha}_h^{-1} \cap B$, and define

$$M_k^B(\mathfrak{M},\chi;h) := \ \{\phi_h \colon \mathcal{H}^r \longrightarrow W_k(\mathbb{C})\colon\ \phi_h|_k\gamma = \chi_0(\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h)\phi_h \ \text{ for } \gamma \in \mathcal{O}_h^\times\}. \tag{14}$$

Note that $\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h \in \widehat{\mathcal{O}}^\times$, so it makes sense to evaluate χ_0 on it.

Lemma 4.3. *The map*

$$\begin{aligned}\Phi\colon\ M_k^B(\mathfrak{M},\chi) &\longrightarrow \textstyle\bigoplus_{h=1}^H M_k^B(\mathfrak{M},\chi;h)\\ \phi &\longmapsto (\phi_h)_{h\in[H]}\end{aligned}$$

is an isomorphism.

Proof. The map Φ is well-defined because for $\gamma \in \mathcal{O}_h^\times$,

$$(\phi_h|_k\gamma)(z) \ = \frac{(\det\gamma)^{k/2}}{j(\gamma,z)^k}\phi(\gamma z,\hat{\alpha}_h)^\gamma \ = \phi(z,\gamma^{-1}\hat{\alpha}_h) \ = \phi(z,\hat{\alpha}_h(\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h)) \ = \chi(\hat{\alpha}_h^{-1}\gamma^{-1}\hat{\alpha}_h)\phi_h(z).$$

It is an isomorphism because by Definition 4.1, knowing $\phi(z,\hat{\alpha}_h)$ for all $z \in \mathcal{H}^r$ and $h \in [H]$ is enough to recover ϕ on all of $\mathcal{H}^r \times \widehat{B}^\times$. □

We now define Hecke operators on $M_k^B(\mathfrak{N},\chi)$. Given a prime ideal $\mathfrak{p} \subset \mathbb{Z}_F$, let $\hat{\pi} \in \widehat{B}^\times$ be an element which at places $v \neq \mathfrak{p}$ is 1 and at $v = \mathfrak{p}$ is an element whose reduced norm is a uniformizer for $F_\mathfrak{p}$. The choice of $\hat{\pi}$ does not affect the double coset $\widehat{\mathcal{O}}^\times\hat{\pi}\widehat{\mathcal{O}}^\times$, which is all that will matter. Equivalently, we may choose $\hat{\pi}$ such that $\operatorname{nrd}(\hat{\pi})\hat{\mathbb{Z}}_F \cap F = \mathfrak{p}$. Let P be $\operatorname{Nm}(\mathfrak{p})+1$ if $\mathfrak{p} \nmid \mathfrak{N}$ and $\operatorname{Nm}(\mathfrak{p})$ otherwise. There exist elements $\{\hat{\pi}_j\}_{j=1}^P \subset \widehat{B}^\times$ such that

$$\widehat{\mathcal{O}}^\times\backslash\widehat{\mathcal{O}}^\times\hat{\pi}\widehat{\mathcal{O}}^\times \ = \bigsqcup_j \hat{\mathcal{O}}^\times\hat{\pi}_j.$$

For $\phi \in M_k^B(\mathfrak{M},\chi)$, we define the Hecke operator

$$(T_\mathfrak{p}\phi)(z,\hat{x}) := \sum_j \phi(z,\hat{x}\hat{\pi}_j^{-1})\chi_0(\hat{\pi}_j). \tag{15}$$

The right-hand side of Equation (15) is independent of the choices of $\hat{\pi}_j$. We can ask how the Hecke operator interacts with the isomorphism of Lemma 4.3.

Lemma 4.4. *There exist:*

1. *A function $j^*\colon [H] \to [H]$ for every $j \in [P]$;*
2. *Elements*

$$\left\{\varpi_{j,h} \in \hat{\alpha}_{j^*(h)}\hat{\mathcal{O}}^\times\hat{\pi}_j\hat{\alpha}_h^{-1} \ \cap B_+^\times\right\}_{\substack{j\in[P]\\ h\in[H]}},$$

where $\varpi_{j,h}$ is well-defined up to multiplication on the left by $\mathcal{O}_{j^*(h)}^\times$;

such that

$$(T_\mathfrak{p}\phi)_h(z) = \sum_{j=1}^P \chi_0(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)(\phi_{j^*(h)}|_k\varpi_{j,h})(z). \tag{16}$$

Proof. By strong approximation (Theorem 4.2), for all $h \in [H]$ and $j \in [P]$, there exist $\varpi_{j,h} \in B_+^\times$, $j^*(h) \in [H]$, and $\hat{u} \in \widehat{\mathcal{O}}^\times$ such that $\hat{\alpha}_h\hat{\pi}_j^{-1} = \varpi_{j,h}^{-1}\hat{\alpha}_{j^*(h)}\hat{u}$. Applying this to Equation (16),

(17)

$$(T_\mathfrak{p}\phi)_h(z) = \sum_{j=1}^P \phi(z,\varpi_{j,h}^{-1}\hat{\alpha}_{j^*(h)}\hat{u})\chi_0(\hat{\pi}_j) \ = \sum_{j=1}^P \chi(\hat{u}\hat{\pi}_j)(\phi_{j^*(h)}|_k\varpi_{j,h})(z) \ = \sum_{j=1}^P \chi(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)(\phi_{j^*(h)}|_k\varpi_{j,h})(z).$$

If for some j and h we have $\varpi_{j,h}^{-1}\hat{\alpha}_{j^*(h)}\hat{u} = (\varpi'_{j,h})^{-1}\hat{\alpha}_{j^*(h)}\hat{u}'$, then $\varpi'_{j,h} = \gamma\varpi_{j,h}$ for some $\gamma \in \mathcal{O}_{j,h}^\times$. By Equation (14), replacing $\varpi_{j,h}$ with $\varpi'_{j,h} = \gamma\varpi_{j,h}$ does not change the summand in Equation (17). □

[DV13, Equation 7.24] describes, in the case of trivial nebentypus, how we can avoid doing computation with adeles by replacing the double coset representatives $\{\hat{\alpha}_h\}_{h\in[H]}$ with the corresponding right $\mathcal{O}$-ideals $\{I_h := \hat{\alpha}_h\hat{\mathcal{O}} \cap B\}$ (representatives of the corresponding right ideal classes in $\mathcal{O}$) and computing the $\varpi_{j,h}$ in terms of these ideals. Similarly, we would like to evaluate the expression $\chi_0(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)$ in Equation (16) without working adelically. By weak approximation [Voi21, Proposition 28.7.3(b)], we may choose $\{\hat{\alpha}_h\}$ such that $(\hat{\alpha_h})_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$ for all $\mathfrak{p} \mid \mathfrak{N}$ and all $h \in [H]$. Then, $\varpi_{j,h} \in \hat{\alpha}_{j^*(h)}\hat{u}\hat{\pi}_j\hat{\alpha}_h^{-1} \in \mathcal{O}$ and

$$\chi_0(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h) \ = \chi_0(\hat{\alpha}_{j^*(h)}^{-1})\chi_0(\varpi_{j,h})\chi_0(\hat{\alpha}_h).$$

With these choices of $\{\hat{\alpha}_h\}$, let $T_\mathfrak{p}$ be the matrix of the Hecke operator on $\bigoplus_h M_k^B(\mathfrak{M},\chi;h)$ defined by Equation (16), and let $T'_\mathfrak{p}$ be the matrix defined by Equation (16) after replacing $\chi_0(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)$ with $\chi_0(\varpi_{j,h})$. Let

$$D := \operatorname{diag}\left(\ldots, \underbrace{\chi_0(\hat{\alpha}_h), \ldots, \chi_0(\hat{\alpha}_h)}_{\dim M_k^B(\mathfrak{M},\chi;h)\text{ times}}, \ldots\right).$$

Then, $T_\mathfrak{p} = DT'_\mathfrak{p}D^{-1}$. Since D is independent of $\mathfrak{p}$, the $\{T'_\mathfrak{p}\}$ and $\{T_\mathfrak{p}\}$ differ only by a change of basis. Therefore, we lose nothing by computing $\{T'_\mathfrak{p}\}$ – avoiding any adelic computation – once we have chosen $\{\hat{\alpha}_h\}$ appropriately.

Let

$$V_k(\mathbb{C}) := \bigotimes_{v\mid\infty}\left(\operatorname{Sym}^{k_v-2}\mathbb{C}^2 \otimes (\det)^{(\underline{k_0}-k_v)/2}\right). \tag{18}$$

When B is definite, $V_k(\mathbb{C}) \cong W_k(\mathbb{C})$, but when B is indefinite, $V_k(\mathbb{C})$ includes factors from the split places that $W_k(\mathbb{C})$ does not.

To apply Equation (16) when B is indefinite, we need to be able to compute Equation (13). In practice, we will only compute $S_k^B(\mathfrak{M},\chi)$ when B is split at exactly one infinite place. By the Eichler-Shimura isomorphism – which generalizes to the setting of Shimura curves and nontrivial nebentypus – for any $h \in [H]$ we have an isomorphism

$$M_k^B(\mathfrak{N},\chi;h) \cong \bigoplus_{h=1}^{H} H^1(\mathcal{O}_h^\times, V_k(\mathbb{C})\otimes\chi). \tag{19}$$

Remark. Readers who are familiar with the usual Eichler-Shimura isomorphism may be surprised that there is only one summand on the right-hand-side of Equation (19). The idea is that by working with $\mathcal{O}_h^\times$ instead of the subgroup $\mathcal{O}_h^1 \subset \mathcal{O}_h^\times$ of norm 1 elements, we only see the holomorphic forms. An analogous example in the setting of modular forms is the isomorphism $M_k(1) \cong H^1(\mathrm{GL}_2(\mathbb{Z}), \operatorname{Sym}^{k-2}\mathbb{C}^2)$. Since we are sweeping some details under the rug anyways, we felt it would ease the presentation to write the isomorphism this way. We refer the reader to [GV11] and [ABB+26] for details.

Remark. Because B is indefinite and we assume that $\mathrm{Cl}_F^+ = 1$, Theorem 4.2 implies that $[H] = 1$. Nonetheless, we describe the Hecke action in slightly more generality than we need to simplify the takeaway later.

Define a Hecke action on $(\varphi_h)_{h\in[H]} \in \bigoplus H^1(\mathcal{O}_h^\times, V_k(\mathbb{C})\otimes\chi)$ by

$$(T_\mathfrak{p}\varphi)_h(\gamma) = \sum_{j=1}^{P}\chi_0(\hat{\alpha}_{j^*(h)}^{-1}\varpi_{j,h}\hat{\alpha}_h)\varphi_{j^*(h)}\left(\varpi_{j,h}\gamma\varpi_{j',j^*(h)}\right)^{\varpi_{j,h}} \tag{20}$$

where the j' in each summand is the unique element of $[P]$ such that $\varpi_{j,h}\gamma\varpi_{j',j^*(h)} \in \mathcal{O}_{j^*(h)}^\times$. With this Hecke action, one can check that Equation (19) is a Hecke module isomorphism [GV11, DV13, ABB+26].

Now that we have defined the Hecke module structure on $M_k^B(\mathfrak{M},\chi)$ and $\bigoplus_h M_k^B(\mathfrak{M},\chi;h)$, we may state the version of the Jacquet-Langlands correspondence that we will use.

Theorem 4.5 (Eichler-Shimizu-Jacquet-Langlands (see e.g. Theorem 3.9 of [DV13]). *Let $\mathfrak{N} \subset \mathbb{Z}_F$ be a level, $k \in \mathbb{Z}_{\geq 2}^n$ a weight, and χ a finite-order Hecke character. Let B/F be a quaternion algebra with discriminant $\mathfrak{D}$ such that $\operatorname{cond}(\chi) | \mathfrak{N}\mathfrak{D}^{-1}$. Then, there is a Hecke module isomorphism*

$$S_k^B(\mathfrak{N}\mathfrak{D}^{-1}, \chi) \cong S_k(\mathfrak{N}, \chi)^{\mathfrak{D}\text{-new}}.$$

When n is odd, we choose B to be an indefinite quaternion algebra ramified at all but one of the infinite places. When n is even, we choose B to be a definite quaternion algebra. In either case, we may take B to be unramified at all finite places, so $\operatorname{disc}(B) = (1)$. For such a B, $S_k^B(\mathfrak{N}, \chi) \cong S_k(\mathfrak{N}, \chi)$ on the nose by Theorem 4.5.

These matrices are not a priori defined over $F'(\chi)$, but Theorem 4.5 and Theorem 2.2 imply that we can always find a basis over which the $\{T_\mathfrak{p}\}$ are defined over $F'(\chi)$. Putting everything together, Theorem 3.1 is proved.

5. Computing spaces of Hilbert modular forms of nonparitious weight

5.1. Fourier coefficients and elemental Hecke operators.

In many settings, the ideal coefficients $\{a_\mathfrak{n}\}$ are the more intrinsic way to think about the coefficients of a Hilbert modular form.

For example, the L-function associated to f, up to twist, is defined on $\operatorname{Re}(s) > 1$ as

$$L(s, f) = \sum_{\mathfrak{n} \subset \mathcal{O}_F} a_\mathfrak{n} \operatorname{Nm}(\mathfrak{n})^{-s} = \prod_\mathfrak{p} (1 - a_\mathfrak{p} \operatorname{Nm}(\mathfrak{p})^{-s} + \chi(\mathfrak{p}) \operatorname{Nm}(\mathfrak{p})^{k_0 - 1 - 2s}). \tag{21}$$

Many constructions of Hilbert modular forms (e.g. as Eisenstein series, CM forms, base change forms, etc.) are given naturally as formulas for the $\{a_\mathfrak{p}\}$, which can then be converted into a Fourier expansion using Equation (4). The procedure in Section 3 is in this vein – from the Hecke matrices $\{T_\mathfrak{p}\}$, we produce the Hecke matrices $\{T_\mathfrak{n}\}$, from these extract the ideal coefficients $\{a_\mathfrak{n}(f)\}$ for f ranging over a basis of $S_k(\mathfrak{N}, \chi)$, and from these can recover the Fourier coefficients of a basis.

This all works in paritious weight because of Theorem 2.2, which guarantees that the $T_\mathfrak{p}$ are defined over $F'(\chi)$. When the weight is nonparitious however, this story breaks down because Theorem 2.2 and Proposition 2.4 do not hold. The field of definition of matrices of the Hecke operator $T_\mathfrak{p}$ acting on $S_k(\mathfrak{N}, \chi)$ and (relatedly) the smallest field containing the ideal coefficient $a_\mathfrak{p}(f)$ for a normalized eigenform $f \in S_k(\mathfrak{N}, \chi)$ depend on the prime $\mathfrak{p}$ when k is nonparitious – this field will generally contain $\sqrt{\pi}$ for any totally positive generator π of $\mathfrak{p}$. In particular, $\mathbb{Q}(\{a_\mathfrak{n}(f)\})$ will be infinite. Because in practice we only want to compute finitely many terms of q-expansions, we could address this issue by working in a very large field containing the fields of definition of the finitely many $\{a_\mathfrak{p}\}$ that we want to compute. However, this would be extremely inefficient at high precisions, and would introduce many potential errors when coercing between different number fields. We will take a different approach.

The first observation is that while the ideal coefficients of a nonparitious eigenform are not defined over a number field, the Fourier coefficients will be.

Theorem 5.1 (Proposition 1.3 of [Shi78]). *If $f \in M_k(\mathfrak{N}, \chi)$ is a normalized Hecke eigenform, then $\mathbb{Q}(\{a_\nu(f)\})$ is a finite extension $F'(\chi)$.*

One way to think about this is the square root factors in Equation (4) (which are not generally elements of F^{gal} when k is nonparitious) exactly cancel out the square roots which cause $\mathbb{Q}(\{a_\mathfrak{n}(f)\})$ to be infinite in the first place. Motivated by Theorem 5.1, we will define "elemental" Hecke operators, indexed by totally positive elements of F. These elemental Hecke operators act on $S_k(\mathfrak{N}, \chi)$ via matrices defined over $F'(\chi)$ (Theorem 5.8) As such, we may replace the usual Hecke algebra (which only makes sense over $\overline{\mathbb{Q}}$ in general) with the "elemental" Hecke algebra, the $F'(\chi)$-algebra generated by the elemental Hecke operators. This elemental Hecke algebra has all of the nice properties that the usual Hecke algebra has in the paritious case. Intuitively, just as the usual Hecke algebra realizes the ideal coefficients as eigenvalues, the elemental Hecke algebra realizes the Fourier coefficients as eigenvalues. This will let us prove Theorem 1.1, and along the way, provide alternative proofs of Theorem 2.2, Theorem 5.1, Proposition 2.4, and Proposition 2.5.

5.2. Elemental Hecke operators and the elemental Hecke algebra.

Definition 5.2. Let μ be a totally positive generator for $\mathfrak{m}$. We define the elemental Hecke operator

$$T_\mu := \mu^{(k - \underline{k_0})/2} T_\mathfrak{m}, \tag{22}$$

where $T_\mathfrak{m}$ is defined as in Equation (5).

Pick a totally positive generator δ for the different $\mathfrak{d}_F$. Let f be a normalized $\mathbb{T}$-eigenform. Combining Definition 5.2 with Equation (4),

$$T_\mu f = \mu^{(k-\underline{k_0})/2} a_\mathfrak{m}(f) = \mu^{(k-\underline{k_0})/2}(\delta^{-1}\mu)^{(\underline{k_0}-k)/2} a_{\delta^{-1}\mu} = \delta^{(k-\underline{k_0})/2} a_{\delta^{-1}\mu}.$$

Writing $\tilde{f} := \delta^{(k-\underline{k_0})/2} f$, we see that

$$T_\mu \tilde{f} = a_{\delta^{-1}\mu}(\tilde{f})\tilde{f}.$$

The eigenvalues of T_μ therefore give the coefficient $a_{\delta^{-1}\mu}$ of a scalar multiple of the original normalized eigenform.

Applying Equation (4) to Equation (5), we can produce a formula for the Fourier coefficients of $T_\pi f$ in terms of those of f. Writing $\daleth, \mu$, and α for totally positive generators of $\mathfrak{n}, \mathfrak{m}$, and $\mathfrak{a}$,

$$a_{\delta^{-1}\daleth}(T_\mu f) = \sum_{\mathfrak{n}+\mathfrak{m}\subset\mathfrak{a}} \chi^*(\mathfrak{a})\,\mathrm{Nm}(\mathfrak{a})^{k_0-1}\alpha^{k-\underline{k_0}}\; a_{\delta^{-1}\daleth\mu\alpha^{-2}}(f). \tag{23}$$

Lemma 5.3. *The elemental Hecke operators $\{T_\pi\}$ on $S_k(\mathfrak{N},\chi)$ are defined over $F'(\chi)$ for any $k \in \mathbb{Z}^n_{\geq 2}$.*

Proof. By Theorem 4.5, it suffices to show this result for the Hecke matrices on $S_k^B(\mathfrak{N},\chi)$.

Let $\rho\colon B^\times \to \mathrm{End}(V_k(\mathbb{C}))$ be the representation associated to $V_k(\mathbb{C})$ (Equation (18)). If B is definite, then one can show from Lemma 4.4 that the Hecke operator $T_\mathfrak{p}$ on $S_k^B(\mathfrak{N},\chi)$ acts by an $[H]\times[H]$ block matrix where each block is a linear combination (with coefficients in $\mathbb{Q}(\chi)$) of matrices $\rho(\varpi)$ for various $\varpi \in \{\varpi_{j,h}\}_{\substack{j\in[P]\\ h\in[H]}}$. If B is indefinite, then the situation is more complicated, but again we end up with a block matrix where each block consists of a linear combination (with coefficients in $\mathbb{Q}(\chi)$) of products $\rho(\gamma)\rho(\varpi_{j,h})$, for $\gamma \in B^1$.

As such, the matrices for the action of $T_\mathfrak{p}$ on $M_k^B(\mathfrak{M},\chi)$ can be defined over a field containing the field of definitions of χ, $\rho(\gamma)$ for $\gamma\in B^1$, and $\{\rho(\varpi_{j,h})\}_{j,h}$.

When k is paritious, the exponents $\{\frac{k_0-k_v}{2}\}_{v|\infty}$ in the determinant factors of $V_k(\mathbb{C})$ are integral. Therefore, on $\gamma\in B^\times$, $\bigotimes_v \mathrm{nrd}^{(\underline{k_0}-k_v)/2}$ evaluates to an element of F when k is paritious. Even when k is nonparitious, if $\gamma\in B^1$, then the determinant factors are all trivial. As such, the obstruction to the $\{T_\mathfrak{p}\}$ being defined over a finite extension is the field of definition of $\{\rho(\varpi_{j,h})\}_{j,h}$. and in particular that under $\bigotimes_v \mathrm{nrd}^{(\underline{k_0}-k_v)/2}$, $\{\varpi_{j,h}\}$ will map to some expression involving square roots.

To remedy this, define

$$V_k'(\mathbb{C}) := \bigotimes_{v|\infty} \mathrm{Sym}^{k_v-2}\,\mathbb{C}^2 = V_k(\mathbb{C})\otimes \mathrm{nrd}^{k-\underline{k_0}/2}.$$

By surjectivity of the reduced norm map to $\mathrm{Cl}_F^+ = 1$ (Theorem 4.2), we may choose double coset representatives $\{\hat{\alpha}_h\}_{h=1}^H$ for $B_+^\times\backslash\widehat{B}^\times/\widehat{\mathcal{O}}^\times$ such that $\mathrm{nrd}(\hat{\alpha})\in\hat{\mathbb{Z}}_F^\times$. With these choices in hand, Lemma 4.4 tells us that $\varpi_{j,h}\in\hat{\alpha}_{j^*(h)}\hat{\mathcal{O}}^\times\hat{\pi}_j\hat{\alpha}_l^{-1}\cap B_+^\times$. Therefore, $\mathrm{nrd}(\varpi_{j,h})\in\mathrm{nrd}(\hat{\alpha}_{j^*(h)})\mathrm{nrd}(\hat{\pi}_j)\mathrm{nrd}(\hat{\alpha}_l^{-1})\cap F \;\in\mathfrak{p}$. Lemma 4.4 also tells us we can multiply $\varpi_{j,h}$ on the left by $\mathcal{O}_{j^*(h)}^\times$ without affecting $T_\mathfrak{p}$. Because $\epsilon\in\mathcal{O}_{j^*(h)}^\times$ for any $\epsilon\in\mathbb{Z}_F^\times$ and $\mathrm{Cl}_F^+=1$, we can always realize $T_\mathfrak{p}$ using $\varpi_{j,h}$ such that $\mathrm{nrd}(\varpi_{j,h} = \pi$ for any totally positive generator π of $\mathfrak{p}$. Evaluated on $\{\varpi_{j,h}\}$, the determinant factor in $V_k(\mathbb{C})$ is exactly $\pi^{k_0-k/2}$. It follows that in the basis given by these $\{\hat{\alpha}_l\}$, T_π is given by the formulas in Lemma 4.4 but with the representation $V_k(\mathbb{C})$ replaced by $V_k'(\mathbb{C})$. The point is that by paying the cost of keeping track of the totally positive generator π of $\mathfrak{p}$, we have gotten rid of the determinant factors that were preventing the $\{T_\mathfrak{p}\}$ from being defined over a finite extension.

In particular, for any K over which the representation $V_k'(\mathbb{C})$ of $B^\times$ can be defined, the elemental Hecke operators $\{T_\pi\}$ can be defined over $K(\chi)$. We want to show that the $\{T_\pi\}$ can be defined over $F'(\chi)$.

Start with an Galois extension $K/\mathbb{Q}$ that contains F^{gal} and splits B. The representation $V_k'(\mathbb{C})$ can be certainly defined over K.

Claim 5.4. *For any such K, there exists a subfield $K'\subset K$ such that $V_k'(\mathbb{C})$ can be defined over K' and $K'\cap F^{\mathrm{gal}} = F'$.*

Assuming the claim for now, choose two such extensions K_1 and K_2 such that $K_1 \cap K_2 = F^{\text{gal}}$. By the claim, there exist subextensions K_1' and K_2' such that $K_1' \cap F^{\text{gal}} = K_2' \cap F^{\text{gal}} = F'$. Therefore, $K_1' \cap K_2' = F'$.

By the claim and the preceding discussion, the matrices of the action of the $\{T_\pi\}$ can be defined over $K_1'(\chi)$ and $K_2'(\chi)$. Hence, the characteristic polynomials of $\{T_\pi\}$ – which are independent of the choice of basis – must be defined over $K_1'(\chi) \cap K_2'(\chi) = F'(\chi)$. Because the Hecke algebra is commutative, the characteristic polynomials of $\{T_\pi\}$ being defined over $F'(\chi)$ implies that the matrices $\{T_\pi\}$ can themselves be descended to $F'(\chi)$ [**?**, Chapter 12].

We conclude by sketching a proof of the claim.

Given any automorphism $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$ fixing the weight k, we can define a K-semilinear map $J_\sigma \colon V_k'(\mathbb{C}) \to V_k'(\mathbb{C})$ that commutes with the action of $B^\times$. The maps J_σ are multiplicative in $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$, so they give us a K-semilinear representation of a subgroup $H \subset \operatorname{Gal}(K/\mathbb{Q})$. The fixed points of this semilinear representation are a vector space over the subfield $K' := K^H$. Because H contains lifts of the automorphisms in $\operatorname{Gal}(F^{\text{gal}}/F')$, $K' \cap F^{\text{gal}} = F'$.

□

Remark. We could have proved Theorem 5.8 in other ways. Avoiding Jacquet-Langlands entirely, we could have argued with Hecke operators on $H^n(\Gamma_0(\mathfrak{N}), V_k(\mathbb{C}) \otimes \chi)$, arguing that these operators are defined over $F'(\chi)$ and that the projectors onto the Hecke submodule corresponding to $S_k(\mathfrak{N}, \chi)$ (which is the subspace fixed by the action of matrices of the form $\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}$ for $\epsilon \in \mathbb{Z}_F^\times$) are also defined over $F'(\chi)$. This approach has the advantage of not needing to deal with a field splitting B. We expect that it should be possible to give an argument using coherent cohomology, thinking of Hilbert modular forms as global sections of some automorphic line bundle. This approach has the advantage of treating partial weight one and higher weight forms uniformly. However, incorporating the determinant twists into the line bundle (and in particular, dealing with "square roots") seems subtle, and we elected to avoid these issues.

Applying Equation (22) to Equation (6), we find

$$T_{\pi^t} = T_\pi T_{\pi^{t-1}} - \pi^{k-1}\chi^*(\mathfrak{p}) T_{\pi^{t-2}} \qquad \text{and} \qquad T_{\imath\mu} = T_\imath T_\mu \text{ if } (\imath, \mu) = 1. \tag{24}$$

Definition 5.5. The elemental Hecke algebra, $\mathbb{T}^{\text{elem}} := \mathbb{T}^{\text{elem}}_{F'(\chi)}$, is the $F'(\chi)$ algebra generated by the $\{T_\pi\}$.

When k is paritious, $\mathbb{T}^{\text{elem}}_{F'(\chi)} \cong \mathbb{T}_{F'(\chi)}$, and even when k is nonparitious, $\mathbb{T}^{\text{elem}}_{\overline{\mathbb{Q}}} \cong \mathbb{T}_{\overline{\mathbb{Q}}}$. $\mathbb{T}$-submodules (resp. $\mathbb{T}$-eigenforms) are the same as $\mathbb{T}^{\text{elem}}$-submodules (resp. $\mathbb{T}^{\text{elem}}$-eigenforms).

For a given totally positive generator δ of $\mathfrak{d}_F$, we say that a $\mathbb{T}^{\text{elem}}$-eigenform f is δ-normalized if $a_{\delta^{-1}}(f) = 1$.

Lemma 5.6. *For $\mathbb{T}^{\text{elem}}$ the elementary Hecke algebra on $S_k(\mathfrak{N}, \chi)$ for $k \in \mathbb{Z}^n_{\geq 2}$, there is a bijection*

$$\begin{aligned} \Phi\colon \{\delta\text{-normalized } \mathbb{T}^{\text{elem}} \text{ eigenforms}\} &\longrightarrow \operatorname{Hom}_{F'(\chi)\text{-alg}}(\mathbb{T}^{\text{elem}}, \mathbb{C}) \\ f &\longmapsto (T \longmapsto a_{\delta^{-1}}(Tf)) \end{aligned}.$$

The map $\Phi(f)$ sends T_μ to the coefficient $a_{\delta^{-1}\mu}(f)$, and the field extension of $F'(\chi)$ generated by the image of $\Phi(f)$ is exactly the extension generated by the coefficients of f.

Proof. For a δ-normalized eigenform f, $a_{\delta^{-1}}(T_\mu(f)) = a_{\delta^{-1}\mu}(f)$. The proof of Lemma 5.6 is then the same as that of the analogous fact for classical modular forms. □

This proves finiteness of the extension $\mathbb{Q}(\{a_\nu(f)\})$ for f a δ-normalized eigenform.

Because $\mathbb{T}^{\text{elem}}$ is a finite commutative $F'(\chi)$-algebra, there exist field extensions $K_f/F'(\chi)$ such that $\mathbb{T}^{\text{elem}} \cong \prod_f K_f$. As in Section 3.1, we can write $S_k(\mathfrak{N}, \chi)^{\text{new}} \cong \bigoplus_f V_f$ where each V_f is an irreducible $\mathbb{T}^{\text{elem}}$-submodule on which $\mathbb{T}^{\text{elem}}$ acts by $K_f/F'(\chi)$. Indeed, because $\operatorname{Hom}_{F'(\chi)\text{-alg}}(\mathbb{T}^{\text{elem}}, \mathbb{C})$ is preserved by post-composition with automorphisms of $\mathbb{C}$ fixing $F'(\chi)$, it follows that the sum is indexed by Galois orbits of newforms.

We can then obtain a basis of V_f defined over $F'(\chi)$ exactly as in Section 3.1, by taking the orbit of a trace form $g \in V_f$ under some generator of $\mathbb{T}^{\text{elem}}|_{V_f}$. We can replace Lemma 3.3 with

$$a_\nu(T^j g) = \mathrm{tr}(T^j T_\nu). \tag{25}$$

We want to show that all of $S_k(\mathfrak{N},\chi)$ (not just the new subspace) has a basis over $F'(\chi)$. Applying Equation (4) to Equation (7), and writing ξ for a totally positive generator of $\mathfrak{D}$, we find

$$a_\nu(\iota_{\mathfrak{D}}(f)) = \xi^{(k-\underline{k_0})/2} a_{\nu\xi^{-1}}(f).$$

Because our aim is to produce a basis for $S_k(\mathfrak{N},\chi)$, we do not care about multiplicative factors. As such we can define $\iota'_\xi(f)$ by $a_\nu(\iota'_{\mathfrak{D}}(f)) := a_{\nu\xi^{-1}}(f)$. The image of $\iota'_{\mathfrak{d}} \otimes \mathbb{C}$ is the same as the image of $\iota_{\mathfrak{d}} \otimes \mathbb{C}$, so it does not affect the complex space we produce in the end.

Theorem 5.7. *The space $S_k(\mathfrak{N},\chi)$ has a basis over $F'(\chi)$.*

Proof. First, assume that $k \in \mathbb{Z}^n_{\geq 2}$. We have

$$S_k(\mathfrak{N},\chi) \cong \bigoplus_{\mathfrak{M}|\mathfrak{N}} \bigoplus_{\mathfrak{D}|\mathfrak{N}\mathfrak{M}^{-1}} \bigoplus_{\substack{\text{newform orbits } f \\ \text{of level } \mathfrak{M}}} \iota'_\xi(V_f), \tag{26}$$

where in the summand, ξ is a totally positive generator of $\mathfrak{D}$. As noted above, each V_f has a basis of forms over $F'(\chi)$, and if $g \in V_f$ has coefficients in $F'(\chi)$, so does $\iota'_\xi(g)$.

Suppose instead that k is of partial weight 1. Choose a weight $l \in \mathbb{Z}^n_{\geq 2}$ and nebentypus ψ such that $S_l(\mathfrak{N},\psi)$ contains a set of forms $g_1,\dots,g_d$ with coefficients in $F'(\chi)$ with no common zeroes.

$$\begin{aligned} F\colon\ S_k(\mathfrak{N},\chi) &\longrightarrow S_{k+l}(\mathfrak{N},\chi\psi) \\ f &\longmapsto (fg_1,\dots,fg_d) \end{aligned}$$

We claim that the image of F is exactly the subspace of $h = (h_1,\dots,h_d) \in S_{k+l}(\mathfrak{N},\chi\psi)^d$ where $g_i h_j = g_j h_i$ for all $i,j \in [d]$. For any $f \in S_k(\mathfrak{N},\chi)$, $F(f)$ is in this subspace. Conversely, given h in the subspace, $\frac{h_i}{g_i}$ is independent of i. It is holomorphic because for any $z \in \mathcal{H}$, $v_z(g_i) > 0$, $v_z(g_j) = 0$ by assumption and so $v_z(h_i) = v_z(h_j) + v_z(g_i) \geq v_z(g_i)$.

We already know that $S_{k+l}(\mathfrak{N},\chi\psi)$ has a basis over $F'(\chi)$, and we've just shown that $S_k(\mathfrak{N},\chi)$ can be identified with a subspace of $S_{k+l}(\mathfrak{N},\chi\psi)^d$ cut out by equations in $F'(\chi)$. Therefore, $S_k(\mathfrak{N},\chi)$ also has a basis of q-expansions over $F'(\chi)$. □

Remark. We could have also argued in the partial weight one case assuming that $S_k(\mathfrak{N},\chi)$ is the Hecke stable subspace of a space of modular quotients with a basis over $F'(\chi)$. Since the Hecke operators are given by Equation (23), the Hecke stable subspace also has a basis of forms over $F'(\chi)$. However, as noted in Section 3.2, because Schaeffer's methods have not been generalized to the Hilbert modular setting, this would not give an unconditional proof.

Remark. Theorem 5.7 is tight. The field of coefficients K_f of $f \in S_k(\mathfrak{N},\chi)$ always needs to include the field of definition of χ. Because $a_{\epsilon\nu}(f) = \epsilon^{k/2} a_\nu(f)$, K_f also needs to include $\epsilon^{k/2}$ for $\epsilon \in \mathbb{Z}^\times_{F,>0}$. Picking a unit ϵ such that $\mathbb{Q}(\epsilon) = F$, we deduce that K_f contains F'. As discussed earlier, totally positive units in F are squares since $\mathrm{Cl}^+_F \cong \mathrm{Cl}_F$. s such, $\epsilon^{k/2}$ is an element of F – otherwise, K_f would need to include square roots of some totally positive units.

Theorem 5.8. *The elemental Hecke operators $\{T_\pi\}$ on $S_k(\mathfrak{N},\chi)$ are defined over $F'(\chi)$ for any $k \in \mathbb{Z}^n_{\geq 1}$.*

Proof. If $k \in \mathbb{Z}^n_{\geq 2}$, the result follows from Lemma 5.3. Otherwise, we still know that $S_k(\mathfrak{N},\chi)$ has a basis of forms with coefficients in $F'(\chi)$ by Theorem 5.7. The elemental Hecke operators on $S_k(\mathfrak{N},\chi)$ are given by Equation (23), and in particular $T_\pi f$ has coefficients in $F'(\chi)$ if f does $F'(\chi)$. It follows that in this basis, T_π is given by a matrix with entries in $F'(\chi)$. □

From this and Lemma 5.6, we can use the arguments that we used in the $\mathbb{Z}^n_{\geq 2}$ case to deduce the following theorems.

Proposition 5.9. *Let f be a δ-normalized $\mathbb{T}^{\mathrm{elem}}$ eigenform in $S_k(\mathfrak{N},\chi)$ for any $k \in \mathbb{Z}^n_{\geq 1}$. The extension $\mathbb{Q}(\{a_\nu(f)\})$ is finite.*

Proposition 5.10. *If $f \in S_k(\mathfrak{N}, \chi)$ with $k \in \mathbb{Z}^n_{\geq 1}$ and τ is an automorphism of $\mathbb{C}$ fixing $F'(\chi)$, then*

$$^\tau f(z) := \sum_{\nu \in \mathfrak{d}^{-1}_{F,>0}} \tau(a_\nu(f)) \exp\left(2\pi i \sum_j \iota(\nu_j) z_j\right)$$

is an element in $S_k(\mathfrak{N}, \chi)$. Furthermore, if f is a Hecke eigenform, then so is $^\tau f$.

Observe that Theorem 5.8, Proposition 5.9, and Proposition 5.10 are strict strengthenings of Theorem 2.2, Proposition 2.4, and Proposition 2.5.

5.3. **Computing spaces of forms in nonparitious weight.** With the theory of Section 5.2 in hand, we can compute spaces of nonparitious forms with $k \in \mathbb{Z}^n_{\geq 2}$ by replacing $T_\mathfrak{p}$ with T_π (for some totally positive generator π of $\mathfrak{p}$) and $a_\mathfrak{n}$ with a_ν everywhere. By doing this, we are able to work with spaces and matrices over $F'(\chi)$ instead of dealing with field extensions depending on $\mathfrak{p}$. We walk through the steps of Section 3 and highlight the modifications that need to be made.

1. **Compute "full" Hecke matrices:** To compute T_π, we repeat the procedure in Section 4, choosing $\{\varpi_{j,h}\}$ in Lemma 4.4 whose norms are all equal to π and replacing $V_k(\mathbb{C})$ with $V'_k(\mathbb{C})$ (i.e. forgetting about the determinant factors). The determinant factor in $V_k(\mathbb{C})$ for these $\{\varpi_{j,h}\}$ is simply a twist by $\pi^{k_0 - k/2}$ (independent of j and h). As $T_\mathfrak{p} = \pi^{k_0 - k/2} T_\pi$, removing the determinant factors lets us produce a matrix for T_π.
2. **Restrict the Hecke matrices to Galois orbits of newforms:** Because each T_π is a rescaling of $T_\mathfrak{p}$, we can replace $T_\mathfrak{p}$ with T_π in Proposition 3.2 without changing the subspace we produce. As discussed in Section 5.2, the elemental Hecke algebra still decomposes as a product of fields K_f, so Equation (9) still holds. Therefore, we can decompose the new subspace into kernel of factors of the characteristic polynomial of any generator $T \in \mathbb{T}^{\text{elem}}|_{S_k(\mathfrak{N},\chi)^{\text{new}}}$.
3. **Produce matrices for T_μ from the matrices for T_π:** We can compute these efficiently using Equation (24) and dynamic programming.
4. **Compute q-expansions of an $F'(\chi)$-basis of each newform orbit:** This was discussed in the proof of Theorem 5.7. The key point is that we use Equation (25) in lieu of Lemma 3.3.
5. **Assemble the bases of newform orbits to produce a basis for $S_k(\mathfrak{N}, \chi)$:** This was discussed in the proof of Theorem 5.7, and can be done using Equation (26).

In the case of partial weight one, the methods of Section 3.2 can be applied almost verbatim. We are able to compute bases of spaces in weights $\mathbb{Z}^n_{\geq 2}$ as just described, and compute Hecke operators on the space of modular quotients using Equation (23) instead of Equation (5).

6. Examples

The codebase can be found here.

For now, we leave the reader with the following two examples.

6.1. **A weight $(4,3)$ space over $\mathbb{Q}(\sqrt{2})$.** In this subsection, we reproduce a computation of [DLP19].

Let $k := (4,3)$, $\mathfrak{N} = (\sqrt{2}+3)|7$, and χ the nontrivial ray class character unramified away from $\mathfrak{N}$ and the infinite places. Then, $S_k(\mathfrak{N}, \chi)$ is two-dimensional. Choosing the generator $\delta := -2\sqrt{2}+4$, the coefficients of one of the two conjugate δ-normalized eigenforms in this space are given in Table 1. The coefficients of f lie in $K = \mathbb{Q}[x]/(x^4 + 24x^2 + 46)$.

We give a snippet of the code that we used to compute Table 1 – it runs in under 7 seconds on a single core of an 11^{th} Gen Intel(R) Core(TM) i7-11800H @ 2.30 GHz (my laptop). Of course, there is a lot going on under the hood! Nonetheless, we feel that one of the merits of this work is that it makes such computations accessible even to users who do not wish to get their hands dirty.

```
// specify the field, level, weight, and nebentypus
F := QuadraticField(2);
ZF := Integers(F);
k := [4, 3];
N := Factorization(7*ZF)[1][1];
```

TABLE 1. Some Fourier coefficients of an eigenform in $S_k(\mathfrak{N},\chi)$ for k, $\mathfrak{N}$, and χ given above. Here, π is a totally positive generator of a prime ideal $\mathfrak{p}$, δ is the chosen generator for the different of $\mathbb{Z}_F$, and α is a generator for the coefficient field $K=\mathbb{Q}[x]/(x^4+24x^2+46)$ such that $\alpha^2=7\sqrt{2}-12$.

$\mathrm{Nm}(\mathfrak{p})$	π	$\mathbf{a}_{\pi\delta^{-1}}(\mathbf{f})$
2	$-\sqrt{2}+2$	$-\alpha$
7	$-3\sqrt{2}+5$	$\frac{1}{7}(-2\alpha^3+6\alpha^2-31\alpha-5)$
7	$-\sqrt{2}+3$	$\frac{1}{7}(\alpha^3+26\alpha)$
9	3	$\frac{1}{7}(3\alpha^3+106\alpha)$
17	$-2\sqrt{2}+5$	$\frac{1}{7}(-8\alpha^2-222)$
17	$-4\sqrt{2}+7$	$\frac{1}{7}(3\alpha^3+120\alpha)$
23	$-\sqrt{2}+5$	$\frac{1}{7}(26\alpha^2+564)$
23	$-7\sqrt{2}+11$	$\frac{1}{7}(36\alpha^2+26)$
25	5	$\frac{1}{7}(2\alpha^3-74\alpha)$
31	$-5\sqrt{2}+9$	$\frac{1}{7}(13\alpha^3+30\alpha)$
31	$-3\sqrt{2}+7$	$\frac{1}{7}(-30\alpha^2-122)$
41	$-8\sqrt{2}+13$	$\frac{1}{7}(-90\alpha^2-562)$
41	$-2\sqrt{2}+7$	$\frac{1}{7}(-6\alpha^3-338\alpha)$
47	$-11\sqrt{2}+17$	$\frac{1}{7}(122\alpha^2+78)$
47	$-\sqrt{2}+7$	$-9\alpha^3-164\alpha$
71	$-7\sqrt{2}+13$	$\frac{1}{7}(-74\alpha^3-930\alpha)$
71	$-5\sqrt{2}+11$	$\frac{1}{7}(-2\alpha^3-530\alpha)$
73	$-2\sqrt{2}+9$	$24\alpha^2+302$
73	$-12\sqrt{2}+19$	$\frac{1}{7}(-27\alpha^3-198\alpha)$
79	$-15\sqrt{2}+23$	$\frac{1}{7}(-46\alpha^3-132\alpha)$
79	$-\sqrt{2}+9$	$\frac{1}{7}(47\alpha^3+942\alpha)$
89	$-4\sqrt{2}+11$	$\frac{1}{7}(-206\alpha^2-2262)$
89	$-10\sqrt{2}+17$	$\frac{1}{7}(65\alpha^3+1228\alpha)$
97	$-8\sqrt{2}+15$	$\frac{1}{7}(-234\alpha^2+734)$
97	$-6\sqrt{2}+13$	$\frac{1}{7}(51\alpha^3+1550\alpha)$
103	$-13\sqrt{2}+21$	$\frac{1}{7}(150\alpha^2+1296)$
103	$-3\sqrt{2}+11$	$\frac{1}{7}(114\alpha^3+2110\alpha)$
113	$-16\sqrt{2}+25$	$\frac{1}{7}(46\alpha^3+160\alpha)$
113	$-2\sqrt{2}+11$	$\frac{1}{7}(-10\alpha^3-316\alpha)$
121	11	$\frac{1}{7}(170\alpha^2+4602)$
127	$-7\sqrt{2}+15$	$\frac{1}{7}(96\alpha^2+130)$
127	$-9\sqrt{2}+17$	$\frac{1}{7}(-272\alpha^2-660)$
137	$-4\sqrt{2}+13$	$\frac{1}{7}(-10\alpha^2-50)$
137	$-14\sqrt{2}+23$	$\frac{1}{7}(-74\alpha^2-90)$
151	$-3\sqrt{2}+13$	$\frac{1}{7}(172\alpha^2-162)$
151	$-17\sqrt{2}+27$	$\frac{1}{7}(114\alpha^2-1404)$
167	$-23\sqrt{2}+35$	$\frac{1}{7}(172\alpha^3+902\alpha)$
167	$-\sqrt{2}+13$	$\frac{1}{7}(-398\alpha^2-4944)$
169	13	$\frac{1}{7}(-22\alpha^3-1006\alpha)$
191	$-13\sqrt{2}+23$	$\frac{1}{7}(11\alpha^3+216\alpha)$
191	$-7\sqrt{2}+17$	$10\alpha^3+76\alpha$
193	$-4\sqrt{2}+15$	$-43\alpha^3-632\alpha$
193	$-18\sqrt{2}+29$	$\frac{1}{7}(129\alpha^3+2682\alpha)$
199	$-9\sqrt{2}+19$	$\frac{1}{7}(62\alpha^2-1440)$
199	$-11\sqrt{2}+21$	$\frac{1}{7}(142\alpha^3+682\alpha)$

```
H := HeckeCharacterGroup(N, [1,2]);
chi := H.1;

// controls how many coefficients we compute
BOUND := 200;

// set up the relevant space of Hilbert modular forms
M := GradedRingOfHMFs(F, BOUND);
Mk := HMFSpace(M, N, k, chi);

// compute an basis of q-expansions spanning the space
// the basis will be over F'(chi) = F
Sk := CuspFormBasis(Mk);

// the dimension of the cusp space is 2
assert #Sk eq 2;

// diagonalize the Hecke action to produce an eigenbasis
eigs := Eigenbasis(Mk, Sk : P:=10);
```

6.2. **Weight** $(1,2)$ **forms over** $\mathbb{Q}(\sqrt{2})$ **and** $\mathbb{Q}(\sqrt{5})$**.** This project was motivated by a proposal ([Cal21]) for computing explicit examples of 4-folds of Mumford's type by computing Hilbert modular forms that are expected to be associated to them. These Hilbert modular forms are expected to be non-CM forms of weight $(1,1,2)$ and have coefficients satisfying certain rationality conditions. One also expects a similar geometric origin for non-CM forms of weight $(1,2)$ with coefficients satisfying similar rationality conditions. While do we not yet have any examples of non-CM forms, we report on some of our findings thusfar.

Given k, $\mathfrak{N}$, and χ, we can compute the space of CM forms $D_k(\mathfrak{N},\chi) \subseteq S_k(\mathfrak{N},\chi)$ by searching through the CM extensions of F with conductor dividing $\mathfrak{N}$ and looking for Hecke characters with an infinity type dependent on k and behavior at finite places determined by χ. Given such a Hecke character, one can produce explicit formulas for the Fourier coefficients of the corresponding automorphically induced Hilbert modular form. As such, we can verify whether a given form is CM or not by checking to see if it lands in the CM space.

Theorem 6.1. *The weight* $(1,2)$ *forms over* $F = \mathbb{Q}(\sqrt{2})$ *and* $F = \mathbb{Q}(\sqrt{5})$ *of Galois stable level* $\mathfrak{N}$ *with* $\mathrm{Nm}(\mathfrak{N}) \leq 1500$ *and quadratic nebentypus character are all CM forms.*

It may seem odd to focus on Galois stable level. While there are indications that forms of Galois stable level might relate more easily to geometry, the reason in our setting is practical. For such levels $\mathfrak{N}$, one can check by a class field theory computation that unless $\mathfrak{N}|(2)$, there will always exist Eisenstein series $g \in M_{\underline{1}}(\mathfrak{N},\psi)$ which are nonvanishing at the cusp at infinity. While the Hecke stability method will work with Eisenstein series of any weight, the bottleneck in the computation of $(1,2)$ forms is computing a basis of q-expansions for a space of weight $(1+k_g, 2+k_g)$, where k_g is the weight of the Eisenstein series.

References

[ABB+26] Eran Assaf, Angelica Babei, Ben Breen, Sara Chari, Edgar Costa, Juanita Duque-Rosero, Aleksander Horawa, Jean Kieffer, Avinash Kulkarni, Grant Molnar, Abhijit S. Mudigonda, Michael Musty, Sam Schiavone, Shikhin Sethi, Samuel Tripp, and John Voight. Computing with Fourier expansions of Hilbert modular forms. page in preparation, 2026.

[BG14] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and Galois representations. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 135–187. Cambridge Univ. Press, Cambridge, 2014.

[Buz12] Kevin Buzzard. An example of a non-paritious Hilbert modular form, 2012.

[Cal21] Frank Calegari. Polymath proposal: 4-folds of Mumford's type, 2021.

[DD08] Lassina Dembélé and Steve Donnelly. Computing Hilbert modular forms over fields with nontrivial class group. In *Algorithmic number theory*, volume 5011 of *Lecture Notes in Comput. Sci.*, pages 371–386. Springer, Berlin, 2008.

[Dem07] Lassina Dembélé. Quaternionic Manin symbols, Brandt matrices, and Hilbert modular forms. *Math. Comp.*, 76(258):1039–1057, 2007.

[DK21] Samit Dasgupta and Mahesh Kakde. On constant terms of Eisenstein series. *Acta Arith.*, 200(2):119–147, 2021.

[DLP19] Lassina Dembélé, David Loeffler, and Ariel Pacetti. Non-paritious Hilbert modular forms. *Math. Z.*, 292(1-2):361–385, 2019.

[DS05] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[DV13] Lassina Dembélé and John Voight. Explicit methods for Hilbert modular forms. In *Elliptic curves, Hilbert modular forms and Galois deformations*, Adv. Courses Math. CRM Barcelona, pages 135–198. Birkhäuser/Springer, Basel, 2013.

[DV21] Steve Donnelly and John Voight. A database of Hilbert modular forms. In *Arithmetic geometry, number theory, and computation*, Simons Symp., pages 365–373. Springer, Cham, [2021] ©2021.

[GV11] Matthew Greenberg and John Voight. Computing systems of Hecke eigenvalues associated to Hilbert modular forms. *Math. Comp.*, 80(274):1071–1092, 2011.

[Mar20] Kimball Martin. The basis problem revisited. *Trans. Amer. Math. Soc.*, 373(7):4523–4559, 2020.

[MS15] Richard A. Moy and Joel Specter. There exist non-CM Hilbert modular forms of partial weight 1. *Int. Math. Res. Not. IMRN*, (24):13047–13061, 2015.

[Sch15] George J. Schaeffer. Hecke stability and weight 1 modular forms. *Math. Z.*, 281(1-2):159–191, 2015.

[Shi78] Goro Shimura. The special values of the zeta functions associated with Hilbert modular forms. *Duke Mathematical Journal*, 45(3):637 – 679, 1978.

[Shu] Jerry Shurman. Hecke characters classically and idelically.

[Sno09] Andrew Snowden. On two dimensional weight two odd representations of totally real fields, 2009.

[Voi10] John Voight. Computing automorphic forms on Shimura curves over fields with arbitrary class number. In *Algorithmic number theory*, volume 6197 of *Lecture Notes in Comput. Sci.*, pages 357–371. Springer, Berlin, 2010.

[Voi21] John Voight. *Quaternion algebras*, volume 288 of *Graduate Texts in Mathematics*. Springer, Cham, [2021] ©2021.