Sphere Packing in 8 dimensions

Abhijit Mudigonda

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Abstract

We present Viazovska's proof of [Via17] following the exposition of [Coh17]. Throughout, an effort has been made to keep the treatment modular (pun intended) and somewhat briefer than that of [Coh17]. We will assume the knowledge of an 18.786 student, and omit a review of modular forms and lattices.

Sphere packing is a very natural problem. We may formally define it as follows.

Definition 1. A **sphere packing** is a subset of $\mathbb{R}^n$ that can be decomposed into a union of congruent spheres with disjoint interiors. The **upper density** of a packing P is defined as

$$\limsup_{r\to\infty} \frac{\mathrm{vol}(B_r^n(0)\cap P}{\mathrm{vol}(B_r^n(0)}$$

and the **sphere packing density**, denoted $\Delta_{\mathbb{R}^n}$, is the supremum of the upper densities of all possible packings.

Definition 2. A sphere packing is **periodic** if there exists a lattice such that the packing is invariant under translation by any lattice element. A **lattice packing** is a sphere packing which has a transitive lattice action.

Our main theorem will be the following.

Theorem 3 ([Via17], [Coh+17])**.** $\Delta_{\mathbb{R}^8} = 0.253669508...$ *and can be achieved by a lattice packing corresponding to* E_8.

1 The difficulty of packing spheres

The complexity of both the structure of an optimal packing and the proof of optimality seem to scale quite drastically with dimension. In one-dimension, both problems are trivial. In two dimensions and three dimensions, the most natural packing ("honeycomb" style hexagonal patterns) are optimal and an optimal three-dimensional packing can be constructed by stacking offset layers of two-dimensional packings. *Neither of these are true are we increase the dimension!*

- **There is a "natural" maximal packing.** In two and three dimensions, the most natural packings (hexagonal packings) are optimal.

- **An optimal n-dimensional packing can be constructed by stacking offset layers of $(n-1)$-dimensional packings.** The packing this generates tends to be "not that bad", but it can be shown that recursing this approach would result in a suboptimal packing by the time we reach 10 dimensions.
- **The optimal packing is "crystalline", and is furthermore periodic and lattice.**

The last point is of particular interest - we will define these terms formally in subsequent sections once we introduce lattices, but as it turns out, there are optimal packings in 8 and 24 dimensions that correspond to certain exceptional lattices that exist in these dimensions: the E_8 root lattice and the Leech lattice, respectively. As such, sphere packing in these dimensions is particularly "nice". In general, it is not clear that an "ordered" packing is necessarily optimal.

Given the difficulty of even guessing at the sphere packing density, it is not surprising that *proving* optimality is even harder. The proof in the two-dimensional case is already nontrivial and that in the three-dimensional case [Hal+15] took a lifetime and was only verified using a formal proof system. In this light, the proofs in the case of dimension 8 and 24 are remarkably elegant,

1.1 Bounding the sphere packing density

Given that solving this problem exactly seems so hard, we may ask what the best known bounds on this problem are.

1.1.1 Lower Bounds

Proposition 4.

$$\Delta_{\mathbb{R}^n} \geq \frac{1}{2^n}.$$

Proof. Consider any maximal packing (one which cannot accommodate any further spheres). Then, doubling the radius of every sphere must yield a covering of the entire space since any uncovered point could be the center of a new sphere. Therefore, the density after doubling the radii must be 1, meaning that the original density was at least $\frac{1}{2^n}$ $\square$

One might wonder how good a lower bound we get from the $\mathbb{Z}^n$-packing - where we put a sphere of radius $\frac{1}{2}$ at each point of $\mathbb{Z}^n$. The density of such a packing is just the volume of an n-dimensional ball of radius $\frac{1}{2}$, which is $n^{-O(n)} << 2^{-n}$. Unsurprisingly, the choice of lattice is critical to the success of the approach. One key property is symmetry - highly symmetric lattices tend to yield better sphere packings. Akshay Venkatesh obtained the following result by working with cyclotomic lattices.

Theorem 5 ([Ven12]). *There exists a lattice packing of unit balls of density*

$$\Delta_{\mathbb{R}^n} \geq (\log \log n) n 2^{-n}.$$

for infinitely many values of n.

As we will see, E_8 is a remarkably symmetric lattice.

1.1.2 Upper Bounds

The best known upper bounds for general n, due to Kabatjanskii and Levenshteien, have stayed more or less the same for over 40 years.

Theorem 6 ([KL78])**.**

$$\Delta_{\mathbb{R}^n} \leq 2^{(-0.599\cdots+o(1))n}.$$

Another class of upper bounds, due to Cohn and Elkies, is based on linear programming, although in our presentation the connection to linear programming may not be apparent. These methods are based on finding a "magic function" and exploiting the properties of this magic function to get a sphere packing. Viazovska's breakthrough in [Via17] boils down to finding a magic function that yields a sharp bound in 8 (and later 24 dimensions).

Theorem 7 ([CE03])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Schwartz function and r a positive real number such that $f(0) = \hat{f}(0) > 0, \hat{f}(y) \geq 0$ for all $y \in \mathbb{R}^n$, and $f(x) \leq 0$ for $|x| \geq r$. Then, the sphere packing density in $\mathbb{R}^n$ is at most $\mathrm{vol}(B^n_{r/2})$.*

The proof uses Poisson summation, which we recall here.

Theorem 8 (Poisson summation)**.** *Let Γ be a lattice in $\mathbb{R}^n$, and let f be a Schwartz function. Then,*

$$\sum_{x \in Gamma} f(x+t) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Gamma)} \sum_{y \in \Gamma^*} \hat{f}(y) e^{2\pi i \langle y,t \rangle}.$$

Proof of Theorem 7. Let's start by showing an upper bound over all lattice packings and then generalize to an upper bound over all packings. Let Γ be a lattice. Without loss of generality, we can scale Γ such that the minimum vector length (the minimum length between any two elements of Γ) is r. Then, our lattice packing for Γ corresponds to putting copies of $B^n_{r/2}$ at each vertex, yielding a packing with density

$$\frac{\mathrm{vol}(B^n_{r/2})}{\mathrm{vol}(\mathbb{R}^n/\Gamma)}.$$

It sufffices to show that $\mathrm{vol}(R^n/\Gamma) \geq 1$, which we will do using Poisson summation to exploit the properties of f. Because the minimum vector length of Γ is r and f is nonpositive for vectors with magnitude at least r, we have that

$$\sum_{x \in \Gamma} f(x) \leq f(0).$$

We also know from nonnegativity of $\hat{f}$ that

$$\frac{1}{\mathrm{vol}(\mathbb{R}^n/\Gamma)} \sum_{y \in \Gamma^*} \hat{f}(y) \geq \frac{\hat{f}(0)}{\mathrm{vol}(\mathbb{R}^n/\Gamma)} = \frac{f(0)}{\mathrm{vol}(\mathbb{R}^n/\Gamma)}.$$

Putting these together, we have that

$$\begin{aligned}\sum_{x\in Gamma} f(x) &= \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Gamma)}\sum_{y\in\Gamma^*}\hat{f}(y)\\ \implies f(0) &\leq \frac{f(0)}{\mathrm{vol}(\mathbb{R}^n/\Gamma)}\\ \implies \mathrm{vol}(\mathbb{R}^n/\Gamma) &\geq 1.\end{aligned}$$

We can get arbitrarily close to the density of any packing with a periodic packing by making our repeating set large, so it suffices to show an upper bound over periodic packings. Thus, let's suppose that our the union of the orbits of the vectors $t_1, t_2, \ldots, t_N$. Then, the density of our packing is

$$\frac{N\,\mathrm{vol}(B^n_{r/2})}{\mathrm{vol}(\mathbb{R}^n/\Gamma)}.$$

and we want to bound the volume of the ball below by N. Observe from Theorem 8 that,

$$\sum_{j,k=1}^{N}\sum_{x\in\Gamma} f(t_j - t_k + x) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Gamma)}\sum_{y\in\Gamma^*}\hat{f}(y)\left|\sum_{j=1}^{N} e^{2\pi i\langle y, t_j\rangle}\right|^2.$$

Now we can do more or less the same thing as before. We get an upper bound on the left by keeping only the terms where $j = k$ and $x = 0$ (so we drop everything that doesn't have an $f(0)$) and we get a lower bound on the right by dropping every term except for the one corresponding to $y = 0$. Putting these together, we get the bound

$$Nf(0) \geq \frac{N^2 f(0)}{\mathrm{vol}(\mathbb{R}^n/\Gamma)} \implies \mathrm{vol}(\mathbb{R}^n/\Gamma) \geq N$$

and we're done.

□

A priori it is not clear what the asymptotics of the bounds we get from this method are, but [CZ14] showed that they are no worse than the bounds of Theorem 6. Thus, surprisingly, even though this proof seems incredibly wasteful (we throw away almost all the information of f and $\hat{f}$ except for their values at 0), the bounds still match the best known ones.

Thus, we have turned our problem of finding better upper bounds into a problem of finding functions f that satisfy the conditions of Theorem 7 while minimizing r (the radius beyond which f is nonpositive). Because all of our constraints on f are radial, we can restrict our search to radial functions, turning our optimization problem over multivariate functions into an optimization problem over radial functions. Unfortunately, this restriction is still not good enough to make the optimization tractable. If we restrict further, to functions of the form $f(x) = p(|x|^2)e^{-\pi x^2}$, the problem becomes tractable. Furthermore, it turns out that these functions can approximate arbitrary radial Schwartz functions arbitrarily well. Therefore, *the best upper bound arising from functions in this space can be made arbitrarily close to the best upper bound overall.* We can then plot the best upper bound we get from magic functions in this space along with the best known lower bounds. Observe that we have "equality" in dimensions 8 and 24. This strongly suggests that there exist magic functions with radii $r = \sqrt{2}$ in dimension 8 and $r = 2$ in dimension 24, since this would yield the matching upper bound that we see in Figure 1.1.2

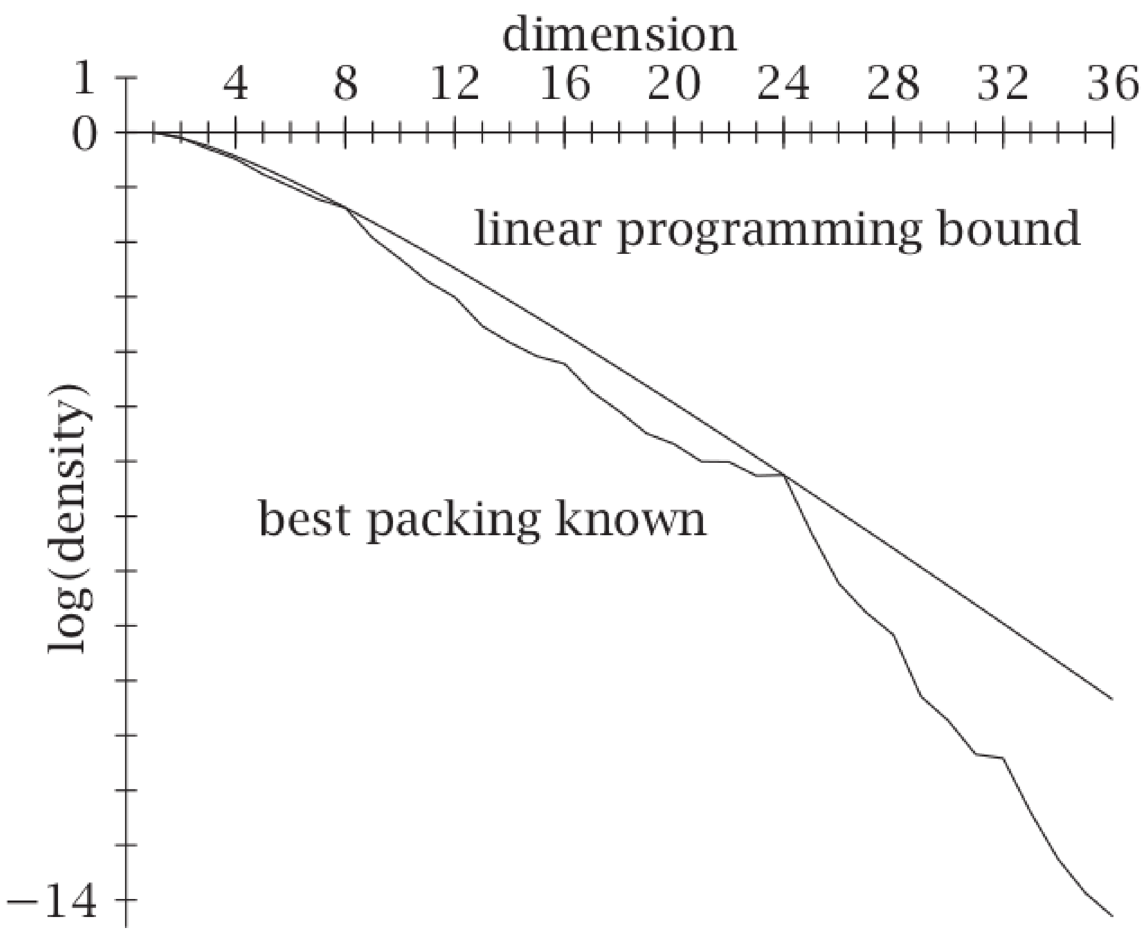

Figure 1: A plot comparing the density of the best known packing to the bound arising from the optimal approximate radial Schwartz function. This image is taken from [Coh17]

2 An upper and lower bound in dimension 8

Definition 9. The E_8 root lattice is the lattice in $\mathbb{R}^8$ with basis vectors having Gram matrix

$$G := \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

To confirm that such a basis does actually exist, we must verify that the Gram matrix is symmetric (immediate) and positive definite.

Fact 10. *The characteristic polynomial of the above matrix is*

$$t^8 - 16t^7 + 105t^6 - 364t^5 + 714t^4 - 784t^3 + 440t^2 - 96r + 1$$

This polynomial has no roots when $t < 0$ because every term is positive and has no root at $t = 0$ so all its eigenvalues are positive.

We will give some properties of the E_8 lattice that will be useful in proving the forthcoming bounds.

Proposition 11. *E_8 is an* ***even, integral, and unimodular*** *lattice.*

Proof. The lattice is integral because the Gram matrix is an integer matrix. It is even because, writing the basis vectors as $\{e_1, \ldots, e_8\}$,

$$|a_1 e_1 + \cdots + a_8 e_8|^2 = \sum_{1 \le i \le 8} 2a_i^2 + \sum_{1 \le i < j \le 8} 2a_i a_j \langle e_i, e_j \rangle \in 2\mathbb{Z}.$$

The volume of the fundamental domain is equal to the absolute value of the determinant of the basis matrix B (columns are basis vectors). However, note that $B^T B = G$ and thus

$$\mathrm{vol}(\mathbb{R}^8/E_8)^2 = \det(B)^2 = \det(B^T)\det(B) = \det(G) = 1$$

where the last equality follows from Fact 10. $\square$

Proposition 12. *Every vector in E_8 has norm of form $\sqrt{2k}$ for $k \ge 0$ and in fact every such norm is represented.*

Next, we will define the dual lattice and show that E_8 is self-dual.

Definition 13. The **dual lattice** Γ^* of a lattice Γ is

$$\{x \in \mathbb{R}^8 \colon \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Gamma\}.$$

Proposition 14. *Every integral unimodular lattice Γ satisfies $\Gamma^* = \Gamma$.*

Now, we're ready to prove some bounds!

2.1 Lower bound

Lemma 15. *The E_8 lattice packing in $\mathbb{R}^8$ has density $\frac{\pi^2}{384}$.*

Proof.

$$\frac{\mathrm{vol}(B^9_{\sqrt{2}/2})}{\mathbb{R}^8/E_8} = \mathrm{vol}(B^9_{\sqrt{2}/2}) = \frac{\pi^2}{384}.$$

□

Now, what remains, and the meat of [Via17], is to show that this is the best we can do.

2.2 Upper bound

We want our upper bound to be $B^n_{\sqrt{2}/2}$ so that it matches the lower bound. Thus, we want a magic function for $r = \sqrt{2}$. Before we even attempt to prove existence, let's start by figuring out what the properties of such an f must be.

Corollary 16. *If f is a magic function with radius r and Γ is a lattice with minimum vector distance r, then the lattice packing corresponding to Γ has density equal to $\mathrm{vol}(B^n_{r/2})$ if and only if $f(\Gamma \setminus \{0\}) = 0$ and $\hat{f}(\Gamma^* \setminus \{0\}) = 0$.*

Proof. Recall from the proof of Theorem 7 the inequality

$$\frac{1}{\mathrm{vol}(\mathbb{R}^n/\Gamma)} f(0) \leq \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Gamma)} \sum_{y \in \Gamma^*} \hat{f}(y) = \sum_{x \in \Gamma} f(x) \leq f(0).$$

Our method yields the desired upper bound if and only if $\mathrm{vol}(\mathbb{R}^n/\Gamma) = 1$, meaning that every inequality above is actually an equality on this lattice, meaning that the egregious dropping of non-origin terms is justified.

□

Now, we can apply Corollary 16 with the lattice E_8 to a magic function f with radius $\sqrt{2}$. Again, we don't know that such an f exists, but we're trying to figure out what properties it must have if it does. By Lemma 14, E_8 is self-dual, and so our condition is that f and $\hat{f}$ vanish at all the lattice points of E_8. Furthermore, Proposition 12 tells us that the points are all at radii of form $\sqrt{2k}$, so it suffices to force f and $\hat{f}$ to have zeros at $r = \sqrt{2k}$ for all $k \geq 1$ (along with the other magic function properties).

Unfortunately, this seems to be quite difficult. Intuitively, the Heisenberg uncertainty principle tells us it is hard to control both a function and its Fourier transform simultaneously. One way to approach this is to write f and $\hat{f}$ in terms of eigenfunctions of the Fourier transform. For a radial function, Fourier inversion tells us that $\hat{\hat{f}} = f$. Thus,

$$f_+ := \frac{f + \hat{f}}{2}$$

and

$$f_- := \frac{f - \hat{f}}{2}$$

are eigenfunctions of the Fourier transform with eigenvalue $+1$ and -1 respectively. Furthermore, if f and $\hat{f}$ share a root set, f_+ and f_- will also share this root set. Thus, we've reduced our problem to constructing radial eigenfunctions of the Fourier transform with the prescribed roots with eigenvalue $+1$ and -1 respectively. Again, we don't actually know that at the end of this we will end up with something satisfying the original conditions - we just know that the actual answer must satisfy these conditions.

2.3 Laplace Transforms and Viazovska's Proof

2.3.1 Building eigenfunctions with the Laplace transform

From class, we are familiar with one eigenfunction of the Fourier transform - the Gaussian function $e^{-\pi x^2}$. Let's study functions $f(x)$ that can be written as continuous linear combinations of Gaussians,

$$f(x) = \int_0^\infty e^{-t\pi|x|^2} g(t)dt$$

where $g(t)$ can be thought of as a "weighting function". This expression is a special case of the Laplace transform of g. If g is relatively nice, we can swap the order of the integral and the Fourier transform and write

$$\begin{aligned} \hat{f}(y) &= \int_0^\infty t^{-\frac{n}{2}} e^{-\frac{\pi|y|^2}{t}} g(t)dt \\ &= \int_0^\infty t^{\frac{n}{2}-2} e^{-t\pi|y|^2} g(\frac{1}{t})dt \end{aligned}$$

Thus, we have that f is an eigenfunction of the Fourier transform with eigenvalue λ if

$$g(t) = \lambda t^{\frac{n}{2}-2} g(\frac{1}{t}).$$

Observe that this looks suspiciously like the modular form transformation law. If we write $g = \phi(it)$ then g satisfies the desired transformation law with $\lambda = i^{\frac{n}{2}-2}$ whenever ϕ is a weight $\frac{n}{2} - 2$ modular form, since then

$$\begin{aligned} g(t) &= \phi(it) \\ &= (it)^{\frac{n}{2}-2} \phi\left(\frac{i}{t}\right) \\ &= (it)^{\frac{n}{2}-2} g\left(\frac{1}{t}\right). \end{aligned}$$

Thus, we've reduced the problem of searching for eigenfunctions to the problem of constructing appropriate modular forms. However, there are two issues with this approach. The first is that we can only construct one of the eigenvalues (whatever $i^{\frac{n}{2}-2}$ is). The second is that this method gives us no way of controlling the zeros of the eigenfunctions we construct, so it appears useless as far as constructing magic functions goes.

2.3.2 Viazovska's solution

Viazovska's solution was to search for eigenfunctions of form

$$\sin^2\left(\frac{\pi|x|^2}{2}\right)\int_0^\infty g(t)e^{-\pi|x|^2 t}dt.$$

Observe that the multiplicative factor is nonnegative but is 0 at $\sqrt{2k}$ for all positive integers k. Now, it remains to find values of g that make this an eigenfunction. Unlike before, the computation of the Fourier transform seems much more complicated, but it turns out that - writing the sine function in terms of complex exponentials - it can be done. It turns out that a $+1$ eigenfunction can be obtatined by taking $g(t) = t^2\phi(\frac{i}{t})$ where ϕ is a weakly holomorphic quasimodular form of weight 0 and depth 2, and a -1 eigenfunction can be obtained by taking $g(t) = \phi(it)$, where ϕ is a weakly holomorphic modular form for $\Gamma(2)$ of weight -2 that satisfies the equation

$$\psi(z) = \psi(z+1) + z^2\psi\left(\frac{-1}{z}\right).$$

The remainder of the proof is computation, showing that the most natural choices of such modular forms do indeed work and yield a magic function, and we've omitted it in the interest of brevity.

3 References

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